# Superexponentiation

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The operations of addition, multiplication, and exponentiation are the first three of an infinite sequence of operations. In view of their importance in mathematics and physics, I will call them *basic* operations.

Generalizing from the first three, we see that each basic operation after the first is derived from the previous one according to the following definition:

Let $*$ be a basic operation which is defined on some number $x$, which I will call the *operand*. We can define a new, higher-order operation $**$ by

$$x ** n = x * x * x * \cdots * x,$$

where the number of $x$'s appearing on the right is a positive integer $n$, which I will call the *exponent*.

For example, $x$ multiplied by 4 is equal to $x + x + x + x$. We can denote this, using the definition's symbolism, by $4 + + x$.

I will indicate the next higher order of operation after $**$ by $***$. Thus, 1.4 cubed can be denoted by $1.4 + + + 3$.

## The arrow operation

Using the above definition, let us investigate superexponentiation, the next basic operation after raising to powers. Since "superexponentiation" is such a mouthful, I will call it the arrow operation, and denote it by the vertical arrow symbol first used by Donald E. Knuth [**1**].

As soon as we try to define the arrow operation, a problem arises. The sequence of operations bifurcates: two arrow operations are possible. The reason for this is that exponentiation is not commutative.

To make this clear, let us first examine the previous level. When we create exponentiation from multiplication, the association of the operands makes no difference, since multiplication is commutative. Thus

$$(x(x(x))) = (((x)x)x),$$

and so $x^n$ has only one definition.

In the case of repeated exponentiation, parentheses make a difference:

$$x^{(x^x)} \quad \text{and} \quad (x^x)^x$$

have different values, in general.

I will call the form on the left above, in which operands are added onto the left side of the parentheses, the left mode of superexponentiation, and I will denote it with the symbol $\uparrow$. For example,

$$3 \uparrow 4 = 3^{[3^{(3^3)}]}.$$

Similarly, I define the right mode by the symbol $\downarrow$. Thus

$$3 \downarrow 4 = \left[(3^3)^3\right]^3.$$

Operations of orders higher than arrow may be indicated by conjunctions of arrows in the manner of the definition ($4 \downarrow \downarrow 3 = (4 \downarrow 4) \downarrow 4$, for example) with the added stipulation that the arrows are executed from left to right (that is, $(4 \uparrow 4) \uparrow 4 = 4 \uparrow \downarrow 3$, and $4 \downarrow (4 \downarrow 4) = 4 \downarrow \uparrow 3$). This convention is arbitrary—the arrows could as well have been read right to left.

As long as we restrict ourselves to positive integral operands, all these operations will be defined.

The numbers that result from these higher-order basic operations quickly become huge. For example, $10\uparrow 4$, which also goes under the name "googolplex," is a number so big that it is physically impractical to write down using exponential notation. (It seems that each order of basic operation allows one to write down numbers that are inconveniently large in the notation of the next-lowest level.) The arrow notation thus opens new realms of magnitude [**1**], [**2**], [**3**].

We can note that for any basic operation $*$, $2*2=4$. This assertion is readily proved: since $x*x=x**2$, $2*2=2**2$. The statement follows by induction from $2+2=4$. Also note that for $*$ other than $+$, $x*1=x$ (by definition).

## Arrow functions

Since $x\uparrow n$ and $x\downarrow n$ are defined for $x$ positive real and $n$ positive integral, we have a family of continuous functions. Graphs of the basic functions $f(x)=x\uparrow n$ and $f(x)=x\downarrow n$ for $n=2$, 3, 4, 5, 6, and 7 are shown in FIGURES 1 and 2.

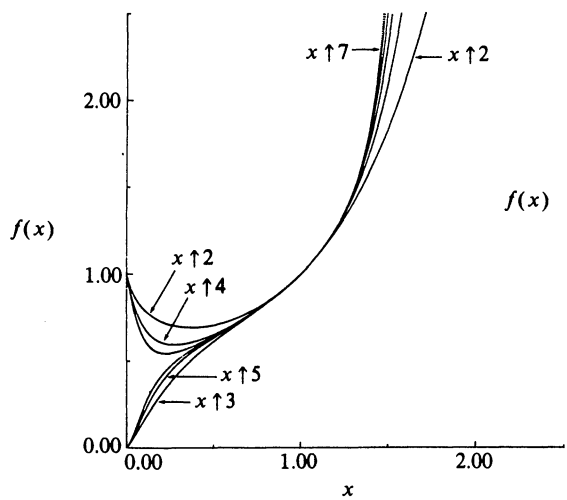


FIGURE 1

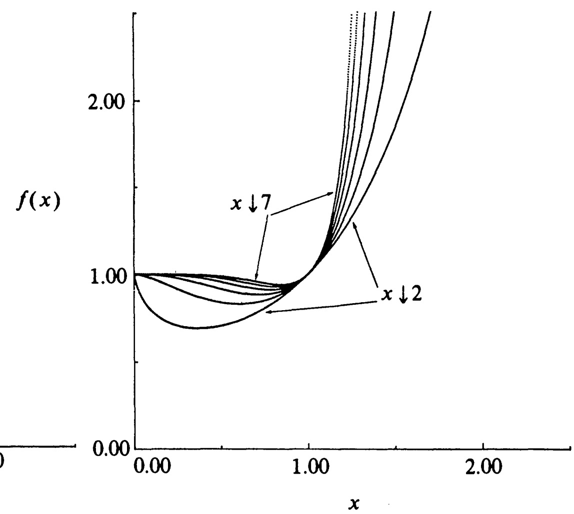


FIGURE 2

Only positive $x$-values are shown because $x^x$ (that is, $x\uparrow 2$ or $x\downarrow 2$) is pathological in the negative $x$ region (and, hence, so are the other functions). It is undefined for irrational $x$, and jumps about among imaginary, positive, and negative values for rational $x$ less than zero.

The "wagging-tail" function $x\uparrow n$ has very interesting behavior as $n$ goes to infinity. Its value quickly goes to infinity for $x$ greater than $e$ to the $1/e$ power (1.44...) and is less than $e$ for $x$ less than $e$ to the $1/e$. For $x$ less than $1/e$ to the $e$ power (0.0659...) it bifurcates into two values which alternate as $n$ is incremented. The value of the function at the bifurcation point is $1/e$. (All of these values were found empirically. Readers may be able to derive them.) To the best of my knowledge, the $x\uparrow n$ function was first investigated in the early 1970's by A. Guyton (private communication).

The bifurcation is shown in FIGURE 3, in which a plot of $f(x)=x\uparrow 1000$ is superimposed on a plot of $g(x)=x\uparrow 1001$. (The two functions are virtually identical to the right of the conjunction point, so that only one line appears there.) The $x$-scale is logarithmic.

## Arrow functions with negative exponents

Single-arrow functions are defined above for positive real operands and whole number exponents. The next step is to try negative integers for exponents. We can do this by figuring out

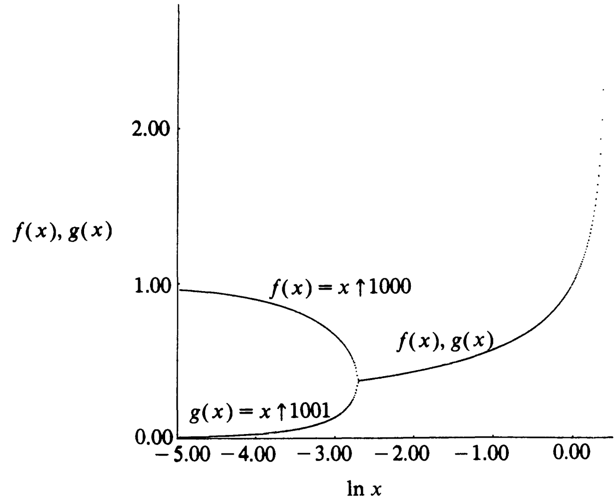


FIGURE 3

a process that will reduce the exponent of an arrow function by one: repeating the process, we will get negative exponents.

To change $x \uparrow n$ to $x \uparrow (n-1)$, we can take the logarithm to the base $x$ of $x \uparrow n$, which gives $x \uparrow (n-1) \log x = x \uparrow (n-1)$. If we do this $n$ times to the function $x \uparrow n$, the result is 1, which we can define as $x \uparrow 0$. Repeating the procedure, we get $x \uparrow (-1) = \log 1 = 0$, and $x \uparrow (-2) = \log 0 =$ minus infinity. It would seem that there is no $x \uparrow n$ for $n$ less than $-2$.

With the $\downarrow$ operation we have more success. The reducing operation here is raising to the $(1/x)$ power: $(x \downarrow n)$ to the $(1/x)$ power is $x \downarrow (n-1)$. This can be applied any number of times if $x$ is positive real.

Some of the negative $\downarrow$ functions are shown in FIGURE 4.

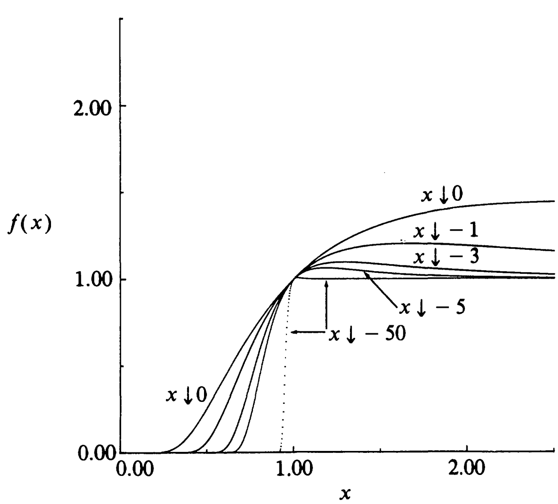


FIGURE 4

## Inverse operations

Once an operation is generated, we can define an inverse operation which "undoes" the work of the operation. I'll say that $\hat{*}$ is the inverse of $*$ if

$$(x \,\hat{*}\, n) * n = x.$$

The number $y = x \,\hat{*}\, n$, where $n$ is a positive integer, I will call "the $n$th asterisk root of $x$" (generalizing the term used for exponentiation's inverse).

For example, $-5$ is the 8th additive root of 3; $1/5$ is the 15th multiplicative root of 3; $1.414\ldots$ is the 2nd exponential root of 2; 3 is the 3rd superexponential (left mode) of 19683; and $1.559\ldots$ is the 2nd superexponential root of 2.

The last-mentioned number is transcendental, by the following argument. If $x^x = 2$, then $x(\ln x) = \ln 2$ and so $x = (\ln 2)/(\ln x)$. Now, the Gelfond-Schneider Theorem [**4**] says that if $y = (\ln a)/(\ln b)$, where $a$ and $b$ are algebraic, then $y$ is either transcendental or rational. So $x$ must be either rational or transcendental. Suppose, for contradiction, that $x$ is rational. Let it equal $p/q$, where $p$ and $q$ are relatively prime (the fraction is in lowest terms). Then $2 = (p/q)^{(p/q)}$ and so

$$p^p = 2^q q^p.$$

By the relative primeness of $p$ and $q$, there are three cases: (1) both $p$ and $q$ are odd, which leads to a contradiction in the above equation (implying one side odd and the other even); (2) $q$ is even and $p$ is odd, which is similarly a contradiction; and (3) $p$ is even and $q$ odd. In this last case, there are integers $m$ and $r$ such that $p = 2^m r$ and $r$ is odd. Equating exponents of 2, we get $mp = q$ or $m = q/p$. But this contradicts the assumption that $p$ and $q$ are relatively prime, since the ratio of relative primes cannot equal an integer. Each case leads to a contradiction. Therefore $x$ is not rational, and so $x$ is transcendental.

As we ascend the orders of basic operations, it seems that every new basic operation's inverse generates roots which are either a new class of reals, or else a new type of number. Subtraction generates negative numbers, division creates fractions, and exponential roots give us algebraic numbers. Exponential roots of negatives create imaginaries.

The preceding two paragraphs lead me to this conjecture: that the arrow operations' inverse operations generate a countable infinity of real superroots, which are, in general, transcendental. I propose to call these superalgebraic numbers.

There are also superroots which are apparently undefined. The equation $x \downarrow 2 = 1/2$, for instance, has no solution in the reals, and seems not to have a complex solution. (I have had no luck with the problematic question of complex solution, but readers may be able to find one, or prove impossibility.) Thus one might say: there is no square superroot of one-half. Historically, this sort of situation has always resulted in the creation of new types of numbers, whose names reflect the idea that they don't make sense ("irrational"), don't exist ("imaginary"), or are simply unpleasant ("negative"). I hope that if anyone ever discovers a consistent way of defining the unreal superroots, he will name them in this tradition.

Note that numbers less than 1 always have two right-mode superroots, and have two left-mode superroots if the root is even.

## Fractional exponents and $\downarrow$

Let's try to get a consistent way to define $x \downarrow (p/q)$, where $p$ and $q$ are positive integers. This would give, in the limit, a definition of $x \downarrow x$ for real $x$, which would immediately give us double-arrow continuous functions $f(x) = x \downarrow \downarrow n$ or $f(x) = x \downarrow \uparrow n$, $n$ a whole number.

Here it is important to proceed by analogy to the established basic operations, lest we be led astray (as I was) by the following formula. It can be shown that all of the positive and negative integer exponent $\downarrow$ functions are generated by

$$x \downarrow p = x^{[x^{(p-1)}]}.$$

This formula gives us values for any fraction $p$, not only integers. Are these values acceptable for defining $x \downarrow \downarrow n$? I now feel they are not, because this definition is not analogous to the way the lower-order operations define fractional exponents.

Let's look at how fractional exponents are handled in multiplication and exponentiation. When multiplying a number by a fraction $p/q$, we multiply by $p$ and then take the $q$th multiplicative root (i.e., divide by $q$). Similarly, to raise a number to a fractional exponential power $p/q$, we raise the number to the $p$th power, and then take the $q$th exponential root.

To continue the process with the arrow operation, we interpret $y = x \downarrow (p/q)$ as $y = (x \downarrow p) \hat{\downarrow}\, q$—and we run into trouble. Here's why: if we raise a number to the superpower $2/3$, it is not the same as raising it to the superpower $4/6$; the two exponents yield different values. (These different values have nothing to do with the fact that there are double roots of numbers less than 1.) This means that the function $y = x \downarrow x$ is neither continuous nor single-valued, which in turn means that the next level of basic continuous functions is undefined.

Similar problems arise with $\uparrow$, with the added complications associated with the "wagging-tail" behavior of $f(x) = x \uparrow n$ for $0 < x < 1$.

The values of $x \downarrow (np/nq)$ seem to converge toward a limit as $n$ goes to infinity (which is not equal to the value given by the formula above for $x \downarrow p$). This raises the interesting possibility that, using a limiting process, fractional exponents of the arrow functions can be defined in a way analogous to lower-level basic operations. To explain: if we pick a number $m$ as the denominator of the exponential fraction, we will get a series of points. As $m$ goes to infinity, the points become a (we hope) smooth curve. This method is difficult to investigate, because the arrow function causes overflow in a computer when only modestly large operands and exponents are inserted.

If the problems of the above program were overcome, perhaps the double-arrow functions could be graphed. Because $(x \downarrow p) \hat{\downarrow} q \neq (x \hat{\downarrow} q) \downarrow p$ in general, there would probably be two classes of such functions.

## Conclusions

A hierarchy of operations on the positive integers can be denoted using the arrow symbol. These operations may be of interest to number theorists, as perfect squares, primes, and so on can be generalized into higher orders of basic operations.

The family of continuous arrow functions is analogous to linear functions at the multiplicative level and to quadratics, cubics, and so on at the exponential level. The arrow functions may find an application in the sciences. Nature uses the first three basic functions profusely: why should she not use arrow?

A new set of numbers, the real superroots, has been conjectured to be transcendental. If a continuous double-arrow function can be developed, another set should result.

It is not clear whether or not the sequence of families of basic continuous functions can be developed beyond the single arrow level. It is clear that more symmetries are lost, and more complications arise, with each new basic operation. While it is unlikely that we will be able to penetrate this mathematical thicket very far, the attempt should be interesting.

**Acknowledgements.** I wish to thank Neal Abraham, Teymour Darkosh, John Lavelle, and Rodica Simion for helpful conversations.

**References**

[1] Donald E. Knuth, Mathematics and computer science: coping with finiteness, Science, 194 (December 1976) 1235–1242. Knuth's notation differs from that used in this paper in using only $\uparrow$, not $\downarrow$, and in using one more arrow (Knuth denotes exponentiation in the manner of the Basic computer language, by $\uparrow$, and left-mode superexponentiation by $\uparrow\uparrow$).

[2] C. Smorynski, Some rapidly growing functions, The Mathematical Intelligencer, 2 (1980) 149–154.

[3] Martin Gardner, Mathematical Games, Scientific American (November 1977) 28. Gardner discusses Knuth's notation.

[4] A. O. Gelfond, Doklady Akad. Nauk S.S.S.R., 2 (1934) 1–6; and Th. Schneider, J. reine angew. Math., 172 (1935) 65–69. The Gelfond-Schneider Theorem is one of Hilbert's 23 problems. See also Ivan Niven, Irrational Numbers, Wiley, chap. 10, or C. L. Siegel and R. Bellman, Transcendental Numbers, Princeton, pp. 80–83.

# Non-Associative Operations

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In many algebraic structures the associativity of an operation is postulated as an axiom. However, there are important non-associative operations (e.g., subtraction, division, vector multiplication) and the purpose of this article is to discuss the extent to which an operation is non-associative.

An operation $*$ defined on a set $S$ is associative only if both ways of inserting parentheses in the product $a_1 * a_2 * a_3$, namely, $(a_1 * a_2) * a_3$ and $a_1 * (a_2 * a_3)$, give the same result for all $a_1$, $a_2$, and $a_3$ in $S$. For some non-associative operations on some sets the non-associativity is incomplete; that is, for products of four or more factors, two different ways of inserting parentheses ('bracketing') give equal results. For example, with the operation of subtraction on the set of real numbers, it is true that $a_1 - (a_2 - (a_3 - a_4)) = (a_1 - (a_2 - a_3)) - a_4$ for any real numbers $a_1$, $a_2$, $a_3$, and $a_4$.

It thus makes sense to say that subtraction of real numbers has "limited" non-associativity. Since one cannot find two ways of bracketing $a_1 - a_2 - a_3$ that lead to the same answer for all $a_1$, $a_2$, and $a_3$, but one can find two bracketings of $a_1 - a_2 - a_3 - a_4$ that always lead to the same result, this leads to a characterization of the non-associativity of subtraction of real numbers as "having depth 4". More generally, the *depth* of non-associativity, $d(*)$, for a non-associative operation on a set $S$ may be defined as:

$$d(*) = \begin{cases} \min\{n > 3\colon & \text{there exist two bracketings of } a_1 * \cdots * a_n \text{ that give the same} \\ & \text{result for every } a_1, \ldots, a_n \text{ in } S\} \\ \infty & \text{if, for each } n \geqslant 3, \text{ there exist elements } a_1, \ldots, a_n \text{ in } S \text{ for which all} \\ & \text{bracketings of } a_1 * \cdots * a_n \text{ give different results.} \end{cases}$$

In the latter case, we will say that $*$ has *unlimited non-associativity*, abbreviated UNA.

For convenience, we shall denote by $N(n)$ the number of ways of inserting parentheses to define unambiguously $a_1 * \cdots * a_n$. So, for example, $N(4) = 5$ corresponding to the bracketings

$$(a_1 * (a_2 * a_3)) * a_4, \quad ((a_1 * a_2) * a_3) * a_4, \quad a_1 * ((a_2 * a_3) * a_4),$$
$$a_1 * (a_2 * (a_3 * a_4)), \quad (a_1 * a_2) * (a_3 * a_4).$$

(For more information on $N(n)$, in particular a derivation of the formula

$$N(n) = \frac{(2n-2)!}{n!(n-1)!},$$

see **[1]** or **[2]**.)