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COMPLETIONS FOR PARTIALLY ORDERED SEMIGROUPS

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ABSTRACT

A standard completion $\mathcal{Y}$ assigns a closure system to each partially ordered set in such a way that the point closures are precisely the (order-theoretical) principal ideals. If S is a partially ordered semigroup such that all left and all right translations are $\mathcal{Y}$-continuous (i.e., $Y \in \mathcal{Y}S$ implies $\{x \in S : y \cdot x \in Y\} \in \mathcal{Y}S$ and $\{x \in S : x \cdot y \in Y\} \in \mathcal{Y}S$ for all $y \in S$), then S is called a $\mathcal{Y}$-semigroup. If S is a $\mathcal{Y}$-semigroup, then the completion $\mathcal{Y}S$ is a complete residuated semigroup, and the canonical principal ideal embedding of S in $\mathcal{Y}S$ is a semigroup homomorphism. We investigate the universal properties of $\mathcal{Y}$-semigroup completions and find that under rather weak conditions on $\mathcal{Y}$, the category of complete residuated semigroups is a reflective subcategory of the category of $\mathcal{Y}$-semigroups. Our results apply, for example, to the Dedekind-MacNeille completion by cuts, but also to certain join-completions associated with so-called "subset systems". Related facts are derived for conditional completions.

0. INTRODUCTION

Various completions of partially ordered semigroups occur sporadically in the literature. Perhaps the best known is the Dedekind-MacNeille completion by cuts (see, for example, [3],[5],[15]), but also various types of "join-completions" are of interest. Some of them are studied in [2],[4],[13],[18],[19],[20],[21]. Both the constructions and the derived results often show a significant similarity; hence it appears now to be high time to furnish the common background and to develop a unifying theory of completions for partially ordered semigroups, including the categorical aspects of such completions. Some of the primary ideas are indicated in the book of Fuchs [15], and the purely

order-theoretic foundations of a general completion theory have been provided to some extent in [9] and [11], so a brief survey on the main facts will suffice for our present purposes (see Section 1).

Let γ be any standard completion, that is, a function assigning to every partially ordered set P (resp. po-semigroup S) a closure system of lower sets that contains all principal ideals. Basic for our investigations will be the category $\mathbf{S}_\gamma$ of γ-semigroups and γ-continuous semigroup homomorphisms, where a map f between posets P and P' is γ-continuous if $f^{-1}[Y'] \in \gamma P$ for all $Y' \in \gamma P'$, and a γ-semigroup is a po-semigroup whose translations are γ-continuous. Apparently, these notions have both topological and order-theoretical applications, and a collection of typical examples is presented at the beginning of Section 2.

Many standard completions γ occurring in practise share the following useful property: every weakly γ-continuous map, that is, every map $f: P \to P'$ such that inverse images of principal ideals belong to γP, is already γ-continuous. This is true, for example, of the MacNeille completion by cuts, the completion by Frink ideals [14], the Aleksandrov completion by lower sets, and the completion by Scott closed sets. Standard completions γ with this property are called compositive, because the class of weakly γ-continuous maps is closed under composition. For any compositive standard completion γ, we can prove that it gives rise to a reflector from the category $\mathbf{S}_\gamma$ to the subcategory $\mathbf{CS}$ of complete residuated semigroups (cl-semigroups in the sense of Birkhoff [3]; see 2.14). If γ fails to be compositive (examples are given in [9] and [11]) then the situation becomes a bit more complicated, and we must replace $\mathbf{CS}$ with the category $\mathbf{S}\vec{\gamma}\mathbf{C}$ of so-called γ-semigroup completions. Its objects(!) are join-dense weakly γ-continuous embeddings $e: S \to C$ of γ-semigroups S in complete residuated semigroups C (these embeddings need not be γ-continuous !). A morphism between two such $\mathbf{S}\vec{\gamma}\mathbf{C}$-objects $e: S \to C$ and $e': S' \to C'$ is then, as one expects, a join-preserving semigroup homomorphism $c: C \to C'$ such that the "restriction" $c_0 = e'^{-1} \circ c \circ e : S \to S'$ is γ-continuous. Our main result in Section 2 is Theorem 2.8, stating that the domain functor from $\mathbf{S}\vec{\gamma}\mathbf{C}$ to $\mathbf{S}_\gamma$ has a left adjoint right inverse $\vec{\gamma}$. On the object level, the "completion" functor $\vec{\gamma}$ assigns to any γ-semigroup S the natural principal ideal embedding

$$\eta_S : S \to \mathcal{Y}S \ , \quad x \mapsto {\downarrow}x = \{ y \in S : y \leqq x \} .$$

In case of a compositive standard completion $\mathcal{Y}$, η_S is the reflection map associated with the reflector $\mathcal{Y}: \mathbf{S}_{\mathcal{Y}} \to \mathbf{CS}$. Hence, η_S or $\mathcal{Y}S$, respectively, may be regarded as the free $\mathcal{Y}$-completion of S.

Almost every standard completion $\mathcal{Y}$ occurring in concrete situations is isomorphism-closed, i.e., every order isomorphism is $\mathcal{Y}$-continuous (of course, this property is much weaker than compositivity). For such completions $\mathcal{Y}$, the objects of the category $\mathbf{S}\vec{\mathcal{Y}}\mathbf{C}$ may be replaced with concrete $\mathcal{Y}$-completions, i.e. pairs (S,C) where C is a complete residuated semigroup and S is a $\mathcal{Y}$-subsemigroup, that is, a $\mathcal{Y}$-semigroup with respect to induced order and multiplication such that the inclusion $S \hookrightarrow C$ is weakly $\mathcal{Y}$-continuous. The resulting adjunction theorem is formulated in 2.9 .

For ordered algebraic structures like groups or cancellative semigroups, etc., one knows that they never can be complete (unless being trivial), so that only conditional completions are of interest. Therefore we study this type of modified completions in Section 3, and it turns out that one has to be very careful when passing from standard completions $\mathcal{Y}$ to their conditional modifications $\mathcal{Y}^o$. It is not hard to see that $\mathcal{Y}^o$ is compositive whenever $\mathcal{Y}$ is (3.3); but the corresponding adjunction theorem (see 3.7) only works when $\mathcal{Y}$ is conditionable, i.e. the inclusion maps $\mathcal{Y}^o P \to \mathcal{Y}P$ are (weakly) $\mathcal{Y}$-continuous - a condition which is violated, for example, if $\mathcal{Y}$ is Frink's ideal completion, while the Dedekind-MacNeille completion by cuts turns out to be conditionable (3.4). Moreover, we can prove that all join-ideal completions arising from so-called subset systems are conditionable (3.8), and these special cases already give a lot of applications, for example to the Aleksandrov completion by lower sets, to the join-completion in the sense of Bishop [4], and to the Scott completion studied by Rosenthal [20]

Some of our investigations are closely related to Aubert's fundamental theory of *x-ideals* [0]. In fact, if $\mathcal{Y}$ is any standard completion and S is a $\mathcal{Y}$-semigroup then $\mathcal{Y}S$ (resp. the corresponding closure operator) is an *x-system* in the sense of Aubert, provided the members of $\mathcal{Y}S$ are invariant under translations. Moreover, the partial order is uniquely determined by $\mathcal{Y}S$. Thanks are due to Professor Bosbach for having called the author's attention to Aubert's paper.

1. STANDARD COMPLETIONS FOR PARTIALLY ORDERED SETS

The theory of standard completions originated with B.Banaschewski [1] and J.Schmidt [21]. A systematic analysis of such completions in a suitable categorical framework has been developed in [2],[9], and [11].

Let us recall some of the basic order-theoretic concepts. For any subset X of a poset P with order relation $\leqq$, define $\downarrow X$, the down set of X, by

$$\downarrow X := \{ y \in P : \text{there exists an } x \in X \text{ with } y \leqq x \}.$$

A subset Y with $Y = \downarrow Y$ is a lower set (or lower end). The sets

$$\downarrow x = \downarrow\{x\}, \quad x \in P,$$

are called principal ideals. Thus, arbitrary unions of principal ideals are lower sets. The collection of all principal ideals of P is denoted by $\mathcal{M}P$, and that of all lower sets by $\mathcal{A}P$. A standard extension $\mathcal{Y}$ is an assignment to each poset P of a collection $\mathcal{Y}P$ of subsets of P with $\mathcal{M}P \subseteq \mathcal{Y}P \subseteq \mathcal{A}P$. If $\mathcal{Y}$ is a standard extension and if each $\mathcal{Y}P$ is a closure system (i.e., closed under arbitrary intersection) then $\mathcal{Y}$ is referred to as a standard completion. The greatest standard completion is the Aleksandrov completion $\mathcal{A}$ by lower sets, and the smallest standard completion is the Dedekind-MacNeille completion or normal completion $\mathcal{N}$, where $\mathcal{N}P$ is the collection of all cuts, i.e., intersections of principal ideals. For each subset X of P,

$$\Delta X = \bigcap \{ \downarrow y : y \in P, \ X \subseteq \downarrow y \}$$

is the cut generated by X, and $\mathcal{N}P = \{ \Delta X : X \subseteq P \}$.

A map $f : P \to P'$ between posets P and P' is said to be isotone if for all $x,y \in P$, $x \leqq y$ implies $f(x) \leqq f(y)$, an embedding if for all $x,y \in P$, $x \leqq y$ is equivalent to $f(x) \leqq f(y)$, and join dense if for all $y' \in P'$, $y' = \sup f[f^{-1}[\downarrow y']]$. Furthermore, $f : P \to P'$ is said to be residuated if there exists a map $g : P' \to P$ such that for all $x \in P$ and $x' \in P'$,

$$f(x) \leqq x' \quad \text{iff} \quad x \leqq g(x').$$

In this case, f and g are isotone maps which uniquely determine each other; f is called the lower (or left) adjoint of g, and g is called the upper (or right) adjoint of f. Notice that f is residuated iff inverse images of principal ideals under f are again prin-

cipal ideals. The following weaker property is of fundamental importance for the study of standard extensions. Given an arbitrary standard extension $\mathcal{Y}$, a function $f: P \to P'$ is said to be <u>weakly $\mathcal{Y}$-continuous</u> if for all $y' \in P'$,

$$f^{-1}[\downarrow y'] \in \mathcal{Y}P ,$$

and <u>$\mathcal{Y}$-continuous</u> if for all $Y' \in \mathcal{Y}P'$,

$$f^{-1}[Y'] \in \mathcal{Y}P .$$

Note that "(weakly) $\mathcal{M}$-continuous" means "residuated" while "(weakly) $\mathcal{A}$-continuous" means "isotone". A residuated map is always weakly $\mathcal{Y}$-continuous, but it need not be $\mathcal{Y}$-continuous! Clearly, all identity maps are $\mathcal{Y}$-continuous and the composition of $\mathcal{Y}$-continuous maps is again $\mathcal{Y}$-continuous. We may thus speak of the category $\mathbf{P}_{\mathcal{Y}}$ of posets and $\mathcal{Y}$-continuous maps.

A standard extension $\mathcal{Y}$ is called <u>compositive</u> if the class of all weakly $\mathcal{Y}$-continuous maps is closed under composition. It has been demonstrated in [9,2.5] that this is tantamount to saying that each weakly $\mathcal{Y}$-continuous map is already $\mathcal{Y}$-continuous. The largest standard completion $\mathcal{A}$ and the smallest standard completion $\mathcal{N}$ are compositive. Many further examples of compositive (and also of non-compositive) standard extensions are to be found in [9]. A slightly weaker condition than compositivity is union completeness. A standard extension $\mathcal{Y}$ is <u>union complete</u> provided $\mathcal{X} \in \mathcal{Y}\mathcal{Y}P$ implies $\bigcup \mathcal{X} \in \mathcal{Y}P$ for each poset P. As observed in [9,2.2], this condition is fulfilled iff the embeddings

$$\eta_P : P \to \mathcal{Y}P , \quad x \mapsto \downarrow x$$

are $\mathcal{Y}$-continuous (η_P is weakly $\mathcal{Y}$-continuous for arbitrary standard extensions $\mathcal{Y}$). Notice that η_P preserves arbitrary meets whereas joins are not, in general, preserved.

If $\mathcal{Y}$ is a standard completion then every subset X of a poset P is contained in a smallest member of $\mathcal{Y}P$, denoted by X^- and called the <u>($\mathcal{Y}$-)closure</u> of X. If X has a join in P then this is also the join of X^- (and conversely), because $\mathcal{Y}P$ contains all principal ideals. Furthermore, $\mathcal{Y}$-continuity of a map $f: P \to P'$ may be characterized, as in the context of topological spaces, by the property

$$f[X^-] \subseteq f[X]^- \qquad (X \subseteq P) .$$

Many standard completions arise as so-called <u>ideal completions</u>. Let Z be an arbitrary assignment to each poset P of a collection ZP of subsets of P (not necessarily lower sets). Then we may define two standard extensions $Z^{\wedge}$ and $Z_{\vee}$ by setting

$$Z^{\wedge}P = MP \cup \{ \downarrow Z : Z \in ZP \},$$

$$Z_{\vee}P = \{ Z \in Z^{\wedge}P : Z \text{ has a join in } P \}.$$

The collection

$$Z^{\vee}P = \{ Y \in AP : \text{ if } Z \subseteq Y \text{ and } Z \in Z_{\vee}P \text{ then } \sup Z \in Y \}$$

defines a standard completion $Z^{\vee}$, the <u>join-ideal completion</u> associated with Z. A related standard completion is the Δ-ideal completion Z^{Δ} (in [9] denoted by $Z^{\sim}$) where

$$Z^{\Delta}P = \{ Y \in AP : \text{ if } Z \subseteq Y \text{ and } Z \in ZP \text{ then } \Delta Z \subseteq Y \}.$$

Obviously, $Z^{\Delta}P$ is always contained in $Z^{\vee}P$, and if P is Z-<u>complete</u>, i.e., each $Z \in ZP$ has a join, then $Z^{\Delta}P$ coincides with $Z^{\vee}P$.

Many authors call an assignment Z of the above type a <u>subset system</u> provided for all isotone maps $f : P \to P'$ and all $Z \in ZP$ the image $f[Z]$ belongs to ZP' (cf.[2],[19],[22]). In order to exclude inconvenient but trivial exceptions, one usually assumes (and we shall do so) that at least one ZP contains a nonempty member. This weak assumption already has the consequence that for each poset P, all singletons $\{x\}$ with $x \in P$ belong to ZP, whence $Z^{\wedge}P = \{ \downarrow Z : Z \in ZP \}$. It follows from [9,4.1] that for any subset system Z of this kind, both $Z^{\vee}$ and Z^{Δ} are <u>compositive</u> standard completions, while $Z^{\wedge}$ is a compositive standard extension iff it is union complete (see also [11]). Three of the most important subset systems are $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{D}$, where

$\mathcal{P}P$ is the collection of all subsets of P (power set),
$\mathcal{F}P$ is the collection of all finite subsets of P, and
$\mathcal{D}P$ is the collection of all directed subsets of P.

(By a <u>directed set</u> we mean a subset D containing an upper bound of each finite subset of D.) By definition,

$\mathcal{P}^{\wedge}P$ is the collection AP of all lower sets,
$\mathcal{F}^{\wedge}P$ consists of all finitely generated lower sets,
$\mathcal{D}^{\wedge}P$ consists of all directed lower sets.

$\mathcal{P}^\wedge$, $\mathcal{F}^\wedge$, and $\mathcal{D}^\wedge$ are compositive, hence union complete, but only $\mathcal{P}^\wedge = \mathcal{A}$ is a standard completion. The join-completion $\mathcal{J} = \mathcal{P}^\vee$ has been studied by various authors, and its categorical aspects have been mentioned in several contexts (see, e.g., [4],[13],[18],[21]). The join-ideal completion $\mathcal{F}^\vee$ frequently occurs in the setting of (join-semi-)lattices, where it coincides with Frink's ideal completion $\mathcal{F}^\Delta$ (see [14]), while $\mathcal{P}^\Delta$ is the normal completion $\mathcal{N}$. $\mathcal{D}^\Delta\mathcal{P}$ and $\mathcal{D}^\vee$ are topological closure systems (in general different, but equal for posets in which every directed subset has a join). The corresponding topologies have been investigated in [7] and [12] (see also [16]). $\mathcal{D}^\vee$ is often referred to as the Scott completion (see, for example, [10],[13] and also [20]). For a map $f: C \to C'$ between two complete lattices C and C', the following four conditions are equivalent:

(a) f is residuated
(b) f is $\mathcal{N}$-continuous
(c) f is $\mathcal{J}$-continuous
(d) f preserves joins, i.e., $f(\sup X) = \sup f[X]$ for all $X \subseteq C$.

More generally, for an arbitrary subset system $\mathcal{Z}$, a map $f: P \to P'$ between posets P and P' is (weakly) $\mathcal{Z}^\vee$-continuous iff it preserves $\mathcal{Z}$-joins, i.e.,

$$Z \in \mathcal{Z}P \text{ and } x = \sup Z \text{ imply } f(x) = \sup f[Z]$$

(see [9,3.3] and [21]).
Now suppose $\mathcal{Y}$ is any standard completion. An important universal property of the maps

$$\eta_P : P \to \mathcal{Y}P, \quad x \to \downarrow x$$

was pointed out first by J.Schmidt [21]; this property plays a crucial role for many adjunction theorems in the theory of partially ordered sets.

1.1 PROPOSITION. (1) *A map $g: P \to C$ from a poset P into a complete lattice C is weakly $\mathcal{Y}$-continuous iff there exists a unique join-preserving map*

$$g^\vee : \mathcal{Y}P \to C \text{ with } g = g^\vee \circ \eta_P, \text{ viz. } g^\vee(Y) = \sup g[Y].$$

(2) *A map $f: P \to P'$ between two posets P and P' is $\mathcal{Y}$-continuous iff there exists a unique join-preserving map $h: \mathcal{Y}P \to \mathcal{Y}P'$ with $h \circ \eta_P = \eta_{P'} \circ f$, viz. $h = \mathcal{Y}f$ with $\mathcal{Y}f(Y) = h[Y]^-$ for all*

$Y \in \mathcal{Y}P$.

$$\begin{tikzcd} P \arrow[d, "\eta_P"'] \arrow[dr, "g"] & \\ \mathcal{Y}P \arrow[r, dashed, "g^\vee"'] & C \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\eta_P"'] & P' \arrow[d, "\eta_{P'}"] \\ \mathcal{Y}P \arrow[r, dashed, "h = \mathcal{Y}f"'] & \mathcal{Y}P' \end{tikzcd}$$

Denoting the category of complete lattices and join-preserving maps by $\mathbf{C}$, we may regard $\mathcal{Y}$ as a functor from the category $\mathbf{P}_{\mathcal{Y}}$ of posets and $\mathcal{Y}$-continuous maps to the category $\mathbf{C}$. If the completion $\mathcal{Y}$ happens to be compositive then $\mathbf{C}$ is a subcategory of $\mathbf{P}_{\mathcal{Y}}$, and Proposition 1.1 shows that the inclusion $\mathbf{C} \hookrightarrow \mathbf{P}_{\mathcal{Y}}$ is a right adjoint to $\mathcal{Y}$, i.e., $\mathbf{C}$ is a reflective subcategory of $\mathbf{P}_{\mathcal{Y}}$, with reflection maps η_P (cf.[9, 4.15] and [11,2.11]). However, for general standard completions $\mathcal{Y}$, the category $\mathbf{C}$ may even fail to be a subcategory of $\mathbf{P}_{\mathcal{Y}}$. An adequate substitute for $\mathbf{C}$ is then the category $\mathbf{P}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$ of $\mathcal{Y}$-<u>completions</u> which is defined as follows: The objects of $\mathbf{P}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$ are weakly $\mathcal{Y}$-continuous join-dense embeddings of posets in complete lattices. A morphism between two $\mathcal{Y}$-completions $e: P \to C$ and $e': P' \to C'$ is a pair (c_0, c), where c is a join-preserving map $c: C \to C'$ and c_0 is a $\mathcal{Y}$-continuous map from P to P' such that $c \circ e = e' \circ c_0$. Notice that in this equation c_0 is uniquely determined by c and vice versa (see [11, Section 3]).

With the help of 1.1 it is not hard to show that the forgetful functor $\overleftarrow{P} : \mathbf{P}_{\mathcal{Y}}^{\rightarrow}\mathbf{C} \to \mathbf{P}_{\mathcal{Y}}$ with

$\overleftarrow{P}e$ = domain of e for $\mathbf{P}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$-objects e and

$\overleftarrow{P}c = c_0$ for $\mathbf{P}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$-morphisms c

is right adjoint to the "completion" functor $\overrightarrow{\mathcal{Y}}: \mathbf{P}_{\mathcal{Y}} \to \mathbf{P}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$ defined by

$\overrightarrow{\mathcal{Y}}P = \eta_P$ for $\mathbf{P}_{\mathcal{Y}}$-objects P and

$\overrightarrow{\mathcal{Y}}f = \mathcal{Y}f$ for $\mathbf{P}_{\mathcal{Y}}$-morphisms f.

This adjunction characterizes the canonical embeddings η_P as <u>free $\mathcal{Y}$-completions</u> in the following sense. For each $\mathcal{Y}$-completion $e: P \to C$ there is a unique join-preserving surjection $e^\vee: \mathcal{Y}P \to C$ such that $e = e^\vee \circ \eta_P$.

For further background on standard extensions, see [9] and [11].

2. COMPLETIONS FOR PARTIALLY ORDERED SEMIGROUPS

We are now going to extend the completion theory alluded to in Section 1 and performed in [11] from posets to partially ordered semigroups. For basic facts and terminology the reader is referred to the monographs [6] and [15]. For lattice-theoretical notions, see [3] and [17].

A po-semigroup (partially ordered semigroup) is a semigroup S equipped with a partial order such that the left translations

$$\ell_x : S \to S \ , \quad y \mapsto x \cdot y \qquad (x \in S)$$

and the right translations

$$r_y : S \to S \ , \quad x \mapsto x \cdot y \qquad (y \in S)$$

are isotone. More generally, given an arbitrary standard extension Y, we call a semigroup together with a partial order a Y-semigroup if left and right translations are Y-continuous (compare the notion of ideal systems in [15,V.17].) Of course, if S is a po-semigroup then we put $YS = YP$ where P is the underlying poset. It will not be necessary to distinguish between P and S.

2.1 EXAMPLES.

(1) An A-semigroup is nothing but a po-semigroup.

(2) M-semigroups are usually referred to as residuated semigroups, in view of the fact that their translations are residuated. Thus a semigroup S equipped with a partial order $\leqq$ is residuated iff there exist binary operations $.^{\cdot}$ and ${}^{\cdot}.$ such that

$$y \leqq z .^{\cdot} x \quad \text{iff} \quad x \cdot y \leqq z \quad \text{iff} \quad x \leqq z {}^{\cdot}. y$$

(cf. [6]). Notice that the residuation operations $.^{\cdot}$ and ${}^{\cdot}.$ are uniquely determined by the multiplication $\cdot$ and the partial order relation $\leqq$. A residuated semigroup is Abelian iff the operations $.^{\cdot}$ and ${}^{\cdot}.$ agree.

(3) If Y is the join-completion $J = P^{\vee}$ (see Section 1) then a Y-semigroup S is characterized by the identities

$$(V) \quad x \cdot \sup Z = \sup \{ x \cdot z : z \in Z \} \text{ and } \sup Z \cdot x = \sup \{ z \cdot x : z \in Z \}$$

for all subsets Z of S possessing a join. This condition is equivalent to:

(W) $x = \sup X$ and $y = \sup Y$ imply $x \cdot y = \sup\{u \cdot v : u \in X, v \in Y\}$.

Similarly, if P_0P denotes, for any poset P, the collection of all nonempty subsets of P then $\mathcal{J}_0 = P_0^{\vee}$ is a join-ideal completion, and a $\mathcal{J}_0$-semigroup is a po-semigroup such that (V) holds for all nonempty subsets Z possessing a join. Partially ordered semigroups with this property are also called <u>lower semicontinuous</u> (cf.[15,XI.7]). Recall that $\mathcal{J}C = \mathcal{M}C$ for every complete lattice C. Hence a complete semigroup is residuated iff (V) holds for arbitrary subsets Z.

(4) The examples in (3) can be generalized in the obvious way. Given an arbitrary subset system Z, it is easy to see that the $Z^{\vee}$-semigroups are those in which the identities (V) are valid for all subsets $Z \in ZS$ which have a join (use the fact that "Z-join preserving" means "$Z^{\vee}$-continuous").

(5) In a <u>join-semilattice</u> S, considered as a po-semigroup, the identities (V) hold for all nonempty subsets possessing a join. Thus every join-semilattice may be regarded as a $\mathcal{J}_0$-semigroup. However, (V) fails for $Z = \emptyset$ unless x is the least element of S.

(6) Now suppose S is a <u>meet-semilattice</u>, considered as a po-semigroup. We discuss four special choices for $\mathcal{Y}$.

(a) S is an $\mathcal{M}$-semigroup iff it is a <u>Brouwerian</u> or <u>relatively pseudo-complemented semilattice</u> (cf.[3],[6]).

(b) S is a $\mathcal{D}^{\wedge}$-semigroup iff it is a <u>distributive semilattice</u>, i.e., for all $x,y,z \in S$ with $x \wedge y \leqq z$ there exist $x_1 \geqq x$ and $y_1 \geqq y$ such that $x_1 \wedge y_1 = z$ (cf.[8],[17]). (Since this equivalence is not completely evident and seems to be widely unknown, we sketch a proof. Suppose S is a $\mathcal{D}^{\wedge}$-semigroup, i.e., each of the sets $M(x,z) = \{y \in S : x \wedge y \leqq z\}$ is a directed (lower) set. Then $x \wedge y \leqq z$ implies $y,z \in M(x,z)$, so we find a $y_1 \geqq y,z$ with $x \wedge y_1 \leqq z$; hence, $x,z \in M(y_1,z)$, and there exists an element $x_1 \geqq x,z$ with $x_1 \wedge y_1 \leqq z$; but then $x_1 \wedge y_1 = z$ because $x_1 \geqq z$ and $y_1 \geqq z$. Conversely, assume S is distributive and choose $y,y' \in M(x,z)$. Then we find $x_1 \geqq x$ and $y_1 \geqq y$ with $x_1 \wedge y_1 = z$. It follows that $z \leqq y_1$ and thus $x \wedge y' \leqq y_1$, so there exist $x_1' \geqq x$ and $y_1' \geqq y'$ such that $x_1' \wedge y_1' = y_1$. Hence y_1' is a member of $M(x,z)$ with $y \leqq y_1'$ and $y' \leqq y_1'$. As z belongs to $M(x,z)$ and $M(x,z)$ is obviously a lower set, we obtain $M(x,z) \in \mathcal{D}^{\wedge}S$, and S is a $\mathcal{D}^{\wedge}$-semigroup.)

(c) S is a $\mathcal{D}^{\vee}$-semigroup iff it is an upper continuous or meet-continuous semilattice, i.e.,

$(\wedge)$ $x \wedge \sup Z = \sup\{x \wedge z : z \in Z\}$

holds for all directed (lower) sets Z possessing a join (cf.[3],[16]).

(d) S is a $\mathcal{P}^{\vee}$-semigroup iff it is a sup-distributive semilattice, i.e., $(\wedge)$ holds for all subsets Z possessing a join (cf.[8]). Complete sup-distributive lattices are also referred to as complete Heyting algebras, frames or locales.

Notice that the following three conditions are equivalent for a complete lattice C (because $\mathcal{M}C = \mathcal{D}^{\wedge}C \cap \mathcal{D}^{\vee}C = \mathcal{P}^{\vee}C$):

(a) C is Brouwerian.
(b) C is distributive and meet-continuous.
(c) C is sup-distributive.

(7) If S is a group equipped with a partial order then the following conditions are all equivalent, because the translations are order-isomorphisms whenever they are isotone:

(a) S is a residuated (semi)group.
(b) S is a $\mathcal{Y}$-semigroup for one, resp. each compositive standard extension $\mathcal{Y}$.
(c) S is a po-(semi)group.

Henceforth let $\mathcal{Y}$ be any standard completion. We can assign to every $\mathcal{Y}$-semigroup S the system $\mathcal{Y}S$, where S is considered as a poset only. For $X,Y \in \mathcal{Y}S$ we set

$$X \cdot Y = \{ z \in S : z = x \cdot y \text{ for some } x \in X, y \in Y \}$$

$$X \odot Y = (X \cdot Y)^{-} = \bigcap \{ Z \in \mathcal{Y}S : X \cdot Y \subseteq Z \}.$$

$\mathcal{Y}$-continuity of the translations yields (cf. Aubert [0]) :

$$X \cdot Y^{-} \subseteq (X \cdot Y)^{-} \quad \text{and} \quad X^{-} \cdot Y \subseteq (X \cdot Y)^{-} \quad \text{for all } X,Y \subseteq S,$$

and it follows that

$$(X \cdot Y)^{-} = (X \cdot Y^{-})^{-} = (X^{-} \cdot Y)^{-} = (X^{-} \cdot Y^{-})^{-} \qquad (X,Y \subseteq S);$$

in particular,

$$X \odot (Y \odot Z) = (X \cdot (Y \cdot Z)^{-})^{-} = (X \cdot (Y \cdot Z))^{-} = ((X \cdot Y) \cdot Z)^{-} = ((X \cdot Y)^{-} \cdot Z)^{-}$$
$$= (X \odot Y) \odot Z$$

for all $X,Y,Z \in \mathcal{Y}S$, and consequently $\mathcal{Y}S$ *is a semigroup with respect to the operation* $\odot$. Furthermore, for $X,Y,Z \in \mathcal{Y}S$ we have the following equivalences:

$$X \odot Y \subseteq Z \text{ iff } X \cdot Y \subseteq Z \text{ iff } \ell_x[Y] \subseteq Z \text{ for all } x \in X$$
$$\text{iff } Y \subseteq \bigcap\{\ell_x^{-1}[Z] : x \in X\} = Z \cdot^{\prime} X \in \mathcal{Y}S .$$

This, together with the opposite calculation for the right translations, leads to

2.2 PROPOSITION. *If S is a $\mathcal{Y}$-semigroup then $\mathcal{Y}S$ is a complete residuated semigroup, and the natural embedding*

$$\eta_S : S \to \mathcal{Y}S , \quad x \mapsto \downarrow x$$

is not only a $\mathcal{Y}$-completion but also a semigroup homomorphism. If S is a residuated semigroup then η_S preserves the residuation operations, too.

2.3 REMARKS.

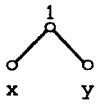
(1) If S is a partially ordered monoid then so is $\mathcal{Y}S$.

(2) Completions of Abelian $\mathcal{Y}$-semigroups are again Abelian.

(3) The completion $\mathcal{Y}S$ of a meet-semilattice S, considered as a $\mathcal{Y}$-semigroup, is again a meet-semilattice. In fact, $\mathcal{Y}S$ is a frame with respect to set inclusion (see 2.1(6)), and the meet-semilattice operation induced from S is set intersection:

$$X \wedge Y = \{x \wedge y : x \in X, y \in Y\} = X \cap Y \text{ for } X,Y \in \mathcal{Y}S .$$

(4) The completion $\mathcal{Y}S$ of a join-semilattice S which is a $\mathcal{Y}$-semigroup need not be a semilattice with respect to the induced operation, although it is a complete lattice with respect to inclusion and set intersection. For example, if $\mathcal{Y}$ is the Aleksandrov completion A then $\circledcirc$ may fail to be idempotent, as the following simple example shows:

$S =$ 

$$A = \{x,y\} \in AS , \quad A \circledcirc A = S \neq A$$

However, if $\mathcal{Y}$ is a standard completion contained in the join-ideal completion $F^{\vee}$ (see Section 1) then $\mathcal{Y}S$ will carry a semilattice operation $\circledcirc$ inherited from S. But, observe that the corresponding partial order relation $\leqq$ defined by

$$X \leqq Y \quad \text{iff} \quad X \circledcirc Y = Y$$

will in general be distinct from set inclusion, since $\emptyset^-$ is not the least but the greatest element with respect to $\leqq$! Indeed,

$$X \circledcirc \emptyset^- = (X \vee \emptyset)^- = \emptyset^- \quad \text{for } X \in \mathcal{Y}S .$$

For example, if $\mathcal{Y}$ is the normal completion $\mathcal{N}$ and S is the preceding three element semilattice then the extended semilattice $(\mathcal{Y}S, \vee)$ has the diagram

while $\mathcal{Y}S$, considered as a poset with respect to set inclusion, has the diagram

These examples show that the interpretation of join-semilattices as po-semigroups is a bit delicate, because the order-inherited natural join-semilattice structure of the completion $\mathcal{Y}S$ is not always the semigroup structure induced from S.
Now let us consider homomorphisms between $\mathcal{J}$-semigroups.

2.4 <u>LEMMA</u>. <u>Let</u> $f: C \to C'$ <u>be a join-preserving map between two</u> $\mathcal{J}$-<u>semigroups</u> C <u>and</u> C' (see 2.1(3)). <u>If there exists a join-dense subset</u> S <u>of</u> C <u>with</u> $f(x \cdot y) = f(x) \cdot f(y)$ <u>for all</u> $x,y \in S$, <u>then</u> f <u>is a semigroup homomorphism</u>.

<u>PROOF</u>. For $x,y \in C$ put $X = \downarrow x \cap S$, $Y = \downarrow y \cap S$. Then we have $x = \sup X$ and $y = \sup Y$, whence $f(x \cdot y) = f(\sup(X \cdot Y)) = \sup f[X \cdot Y] = \sup(f[X] \cdot f[Y])$, and $f(x) \cdot f(y) = f(\sup X) \cdot f(\sup Y) = \sup f[X] \cdot \sup f[Y] = \sup(f[X] \cdot f[Y])$, i.e., $f(x \cdot y) = f(x) \cdot f(y)$. □

Now it is easy to extend Proposition 1.1 to the level of $\mathcal{Y}$-semigroups.

2.5 <u>PROPOSITION</u>. <u>For each</u> $\mathcal{Y}$-<u>semigroup</u> S, <u>the embedding</u> $\eta_S : S \to \mathcal{Y}S$, $x \mapsto \downarrow x$ <u>has the following universal property</u>: <u>A map</u> $g: S \to C$ <u>from</u> S <u>into a complete residuated semigroup</u> C <u>is a weakly</u> $\mathcal{Y}$-<u>continuous semigroup homomorphism iff there exists a unique join-preserving semigroup homomorphism</u> $g^\vee : \mathcal{Y}S \to C$ <u>with</u> $g = g^\vee \circ \eta_S$.

PROOF. Suppose $g: S \to C$ is a weakly $\mathcal{Y}$-continuous semigroup homomorphism from S into a complete residuated semigroup C. By 1.1,

$$g^{\vee} : \mathcal{Y}S \to C \ , \quad Y \mapsto \sup Y$$

is the unique join-preserving map from $\mathcal{Y}S$ into C with $g = g^{\vee} \circ \eta_S$. Furthermore, the restriction of $g^{\vee}$ to $\mathcal{M}S$ satisfies the equation $g^{\vee}(\downarrow x)\cdot g^{\vee}(\downarrow y) = g(x)\cdot g(y) = g(x\cdot y) = g^{\vee}(\downarrow(x\cdot y)) = g^{\vee}(\downarrow x\cdot \downarrow y)$, and as $\mathcal{M}S$ is join-dense in the complete residuated semigroup $\mathcal{Y}S$ (see 2.2), we conclude from Lemma 2.4 that $g^{\vee}$ is a semigroup homomorphism. Conversely, if $h: \mathcal{Y}S \to C$ is any join-preserving semigroup homomorphism with $g = h \circ \eta_S$ then from 1.1 and 2.2 we infer that g must be a weakly $\mathcal{Y}$-continuous semigroup homomorphism (but observe that g need not be $\mathcal{Y}$-continuous if $\mathcal{Y}$ fails to be compositive!) □

2.6 COROLLARY. *A map $f: S \to S'$ between $\mathcal{Y}$-semigroups S and S' is a $\mathcal{Y}$-continuous semigroup homomorphism iff there exists a unique join-preserving semigroup homomorphism $\mathcal{Y}f: \mathcal{Y}S \to \mathcal{Y}S'$ with $\mathcal{Y}f \circ \eta_S = \eta_{S'} \circ f$.*

PROOF. Apply 2.5 to the map $g = \eta_{S'} \circ f$ which is weakly $\mathcal{Y}$-continuous iff f is $\mathcal{Y}$-continuous (cf.[9,2.4]). □

Now let us introduce the adequate categories. The $\mathcal{Y}$-semigroups together with $\mathcal{Y}$-continuous semigroup homomorphisms form a category $\mathbf{S}_{\mathcal{Y}}$. For example, $\mathbf{S}_{\mathcal{A}}$ is the category of po-semigroups and isotone semigroup homomorphisms. By $\mathbf{CS}$ we denote the category of complete residuated semigroups together with join-preserving (i.e., residuated) semigroup homomorphisms. Notice that $\mathbf{CS}$ is a subcategory of $\mathbf{S}_{\mathcal{A}}$ but not necessarily a subcategory of the smaller category $\mathbf{S}_{\mathcal{Y}}$. However, if $\mathcal{Y}$ is compositive then $\mathbf{CS}$ turns out to be a reflective subcategory of $\mathbf{S}_{\mathcal{Y}}$; we shall come back to that situation later. In the general case of an arbitrary standard completion $\mathcal{Y}$ we have to replace the category $\mathbf{CS}$ by another category of so-called "$\mathcal{Y}$-semigroup completions" in order to obtain the desired adjunction theorems.

We start with a somewhat larger category, the category $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$ of $\mathcal{Y}$-semigroup embeddings. The objects of this category are all weakly $\mathcal{Y}$-continuous embeddings of $\mathcal{Y}$-semigroups into complete residuated semigroups. If $e: S \to C$ and $e': S' \to C'$ are two of these $\mathcal{Y}$-semigroup embeddings, then a morphism between e and e' in the category $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$ is a

join-preserving semigroup homomorphism $c: C \to C'$ such that there exists a $\mathcal{Y}$-continuous map $c_0: S \to S'$ with $c \circ e = e' \circ c_0$. The "induced" map c_0 is then an $\mathbf{S}_{\mathcal{Y}}$-morphism and is uniquely determined by c. The category $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$ may be regarded as a generalized "arrow category" (see [18,4.16] and [11]).

By a $\mathcal{Y}$-<u>semigroup completion</u> we mean a $\mathcal{Y}$-semigroup embedding which is join-dense. A $\mathcal{Y}$-semigroup completion $e: S \to C$ is <u>free</u> or <u>primitive</u> if for all $x \in S$ and all $Y \in \mathcal{Y}S$,

(*) $e(x) \leqq \sup e[Y]$ implies $x \in Y$.

By 2.2, each of the natural embeddings η_S is a $\mathcal{Y}$-semigroup completion, and the subsequent lemma will demonstrate that they are, up to isomorphism, the only free $\mathcal{Y}$-semigroup completions. Moreover, we shall characterize (free) $\mathcal{Y}$-completions $e: S \to C$ by means of the unique "$\mathcal{Y}$-liftings" $e^{\vee}: \mathcal{Y}S \to C$ (defined in 2.5).

2.7 <u>LEMMA</u>. <u>Let</u> $e: S \to C$ <u>be a</u> $\mathcal{Y}$-<u>semigroup embedding</u>. <u>Then the following three conditions are equivalent</u>:

(a) e <u>is join-dense, i.e., a</u> $\mathcal{Y}$-<u>semigroup completion</u>.

(b) $e^{\vee}$ <u>is surjective</u>.

(c) <u>There exists a (unique) surjective</u> $\mathbf{CS}$-<u>morphism</u> $c: \mathcal{Y}S \to C$ <u>with</u> $e = c \circ \eta_S$.

<u>Furthermore, the following three conditions are equivalent</u>:

(a') e <u>is a free</u> $\mathcal{Y}$-<u>semigroup completion</u>.

(b') $e^{\vee}$ <u>is bijective, i.e., a</u> $\mathbf{CS}$-<u>isomorphism</u>.

(c') <u>There exists a (unique)</u> $\mathbf{CS}$-<u>isomorphism</u> $c: \mathcal{Y}S \to C$ <u>with</u> $e = c \circ \eta_S$.

<u>PROOF</u>. (a) $\Rightarrow$ (b): For $y \in C$, we have $Y = e^{-1}[\downarrow y] \in \mathcal{Y}S$ and $y = \sup e[e^{-1}[\downarrow y]] = e^{\vee}(Y)$. Hence e is surjective and therefore a $\mathbf{CS}$-epimorphism.

(b) $\Rightarrow$ (c): See 2.5.

(c) $\Rightarrow$ (a): For $y \in C$ there exists an $Y \in \mathcal{Y}S$ with $y = c(Y) = c(\sup \eta_S[Y]) = \sup c[\eta_S[Y]] = \sup e[Y]$. Hence e is join-dense.

(a') $\Rightarrow$ (b'): By what we have shown before, $e^{\vee}$ must be surjective.

Suppose $e^{\vee}(Y) = e^{\vee}(Z)$ for some $Y,Z \in \mathcal{Y}S$. Then, by condition (*), we obtain $x \in Y$ iff $e(x) \leqq e^{\vee}(Y)$ iff $e(x) \leqq e^{\vee}(Z)$ iff $x \in Z$, whence $Y = Z$.

(b') $\Rightarrow$ (c'): See 2.5.

(c') $\Rightarrow$ (a'): Since (c) implies (a), we already know that e must be a $\mathcal{Y}$-semigroup completion. Furthermore, for $x \in S$ and $Y \in \mathcal{Y}S$ with $e(x) \leqq \sup e[Y]$, we conclude $c(\downarrow x) = (c \circ \eta_S)(x) \leqq \sup c[\eta_S[Y]] = c[\sup \eta_S[Y]] = c(Y)$, and as c is an isomorphism, it follows that $\downarrow x \subseteq Y$, i.e. $x \in Y$. $\square$

REMARK. It is easy to see that the epimorphisms (resp. monomorphisms) in the category $\mathbf{C}$ (see Section 1) are precisely the surjective (resp. injective) $\mathbf{C}$-morphisms. But it is not evident whether the same holds for the category $\mathbf{CS}$ instead of $\mathbf{C}$. However, the isomorphisms in $\mathbf{CS}$ are certainly the bijective $\mathbf{CS}$-morphisms.

The $\mathcal{Y}$-semigroup completions are the objects of a full subcategory $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$ of the category $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$ and the free $\mathcal{Y}$-semigroup completions are the objects of a full subcategory $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{F}$. Suppose there is given an $\mathbf{S}_{\mathcal{Y}}$-morphism $f: S \to S'$ and a $\mathcal{Y}$-semigroup embedding $e: S' \to C'$. Then $e \circ f$ is weakly $\mathcal{Y}$-continuous, and by 2.5, the map $(e \circ f)^{\vee}$ is the unique $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$-morphism $c: \eta_S \to e$ with $c_0 = f$. This proves the main part of the following result.

2.8 THEOREM. *Let $\mathcal{Y}$ be an arbitrary standard completion.*

(1) *The domain functor $\overleftarrow{P} : \mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E} \to \mathbf{S}_{\mathcal{Y}}$ with*
$\overleftarrow{P}e$ = *domain of* e *for* $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$*-objects* e *and*
$\overleftarrow{P}c = c_0$ *for* $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$*-morphisms* c
has a left adjoint right inverse $\overrightarrow{\mathcal{Y}} : \mathbf{S}_{\mathcal{Y}} \to \mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{E}$ *with*
$\overrightarrow{\mathcal{Y}}S = \eta_S$ *for* $\mathbf{S}_{\mathcal{Y}}$*-objects* S *and*
$\overrightarrow{\mathcal{Y}}f = \mathcal{Y}f$ *for* $\mathbf{S}_{\mathcal{Y}}$*-morphisms* f.

(2) $\overleftarrow{P}$ *restricts to a forgetful (i.e. faithful) functor from* $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$ *to* $\mathbf{S}_{\mathcal{Y}}$ *which is right adjoint left inverse to the corestriction of the functor* $\overrightarrow{\mathcal{Y}}$ *to* $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{C}$.

(3) $\overleftarrow{P}$ *restricts to an equivalence between the categories* $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{F}$ *and* $\mathbf{S}_{\mathcal{Y}}$; *the inverse equivalence is obtained by corestricting the functor* $\overrightarrow{\mathcal{Y}}$ *to* $\mathbf{S}_{\mathcal{Y}}^{\rightarrow}\mathbf{F}$.

(4) *In each of these adjunctions, the counit* ε *is given by*

$$\varepsilon_e : \mathcal{Y}S \to C \ , \quad Y \mapsto \sup e[Y]$$

for all $\mathcal{Y}$*-semigroup embeddings* $e: S \to C$.

In almost all practical applications, the standard completions $\mathcal{Y}$ are *isomorphism-closed*, i.e., every order-isomorphism is $\mathcal{Y}$-continuous. In this case it is convenient to replace the categories $\mathbf{S}_{\mathcal{Y}}^{\to}\mathbf{E}$, $\mathbf{S}_{\mathcal{Y}}^{\to}\mathbf{C}$ and $\mathbf{S}_{\mathcal{Y}}^{\to}\mathbf{F}$ with certain subcategories of the product category $\mathbf{S}_{\mathcal{Y}} \times \mathbf{CS}$. This can be done as follows: Objects of the category $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$ are pairs (S,C) where C is a complete residuated semigroup and S is a $\mathcal{Y}$-*subsemigroup* of C , i.e., a subsemigroup of C which is a $\mathcal{Y}$-semigroup with respect to the induced partial order relation and a $\mathcal{Y}$-*subposet*, i.e.,

$$\downarrow y \cap S \in \mathcal{Y}S \quad \text{for all} \quad y \in C .$$

In other words, (S,C) is an object of the category $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$ iff the inclusion map $i_{S,C}: S \to C$ is a $\mathcal{Y}$-semigroup embedding. A morphism between two $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$-objects (S,C) and (S',C') is a join-preserving semigroup homomorphism $c: C \to C'$ inducing a $\mathcal{Y}$-continuous map $c_o: S \to S'$ (which is then a semigroup homomorphism between S and S').

There is an obvious equivalence functor $\mathcal{I}: \mathbf{S}_{\mathcal{Y}}\mathbf{E} \to \mathbf{S}_{\mathcal{Y}}^{\to}\mathbf{E}$ with

$\mathcal{I}(S,C) = i_{S,C}$ for $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$-objects (S,C) and

$\mathcal{I}c = c$ for $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$-morphisms c .

This functor induces equivalences between the full subcategory $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$ of *concrete* $\mathcal{Y}$-*semigroup completions* (defined in the obvious way) and the category $\mathbf{S}_{\mathcal{Y}}^{\to}\mathbf{C}$, as well as between the full subcategory $\mathbf{S}_{\mathcal{Y}}\mathbf{F}$ of *free concrete* $\mathcal{Y}$-*semigroup completions* and the category $\mathbf{S}_{\mathcal{Y}}^{\to}\mathbf{F}$.

An $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$-object (S,C) belongs to $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$ iff $y = \sup(\downarrow y \cap S)$ for all $y \in C$. The objects of $\mathbf{S}_{\mathcal{Y}}\mathbf{F}$ are characterized by the additional property that each element of S is $\mathcal{Y}$-*prime* in (S,C), i.e.,

$$x \in S \ , \ Y \in \mathcal{Y}S \ \text{ and } \ x \leqq \sup Y \ \text{ imply } \ x \in Y .$$

Now the "concrete" version of Theorem 2.8 reads as follows:

2.9 THEOREM. *Let* $\mathcal{Y}$ *be any isomorphism-closed standard completion. Then the "left projection" functor* $P_E: \mathbf{S}_{\mathcal{Y}}\mathbf{E} \to \mathbf{S}_{\mathcal{Y}}$ *with*

$P_E(S,C) = S$ *for* $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$*-objects and*

$P_E c = c_o$ *for* $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$*-morphisms* c

is right adjoint to the "$\mathcal{Y}$*-embedding" functor* $\mathcal{Y}_E: \mathbf{S}_{\mathcal{Y}} \to \mathbf{S}_{\mathcal{Y}}\mathbf{E}$ *with*

$\mathcal{Y}_E S = (\mathcal{M}S, \mathcal{Y}S)$ *for* $\mathbf{S}_{\mathcal{Y}}$*-objects* S *and*
$\mathcal{Y}_E f = \mathcal{Y}f$ *for* $\mathbf{S}_{\mathcal{Y}}$*-morphisms* f.

The unit of the adjunction is an isomorphism, and the components of the counit are the join maps

$$\varepsilon_{(S,C)} : \mathcal{Y}S \to C \ , \quad Y \mapsto \sup Y .$$

2.10 COROLLARY. *The adjoint pair* $\mathcal{Y}_E$, $\mathcal{P}_E$ *restricts to an adjoint pair of functors*

$$\mathcal{Y}_C : \mathbf{S}_{\mathcal{Y}} \to \mathbf{S}_{\mathcal{Y}}\mathbf{C} \ , \quad \mathcal{P}_C : \mathbf{S}_{\mathcal{Y}}\mathbf{C} \to \mathbf{S}_{\mathcal{Y}} .$$

The projection functor $\mathcal{P}_C$ *is faithful because the counit maps* $\varepsilon_{(S,C)}$ *are surjective for* $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$*-objects* (S,C). *Furthermore,* $\mathcal{P}_C$ *restricts to an equivalence between* $\mathbf{S}_{\mathcal{Y}}$ *and* $\mathbf{S}_{\mathcal{Y}}\mathbf{F}$; *the inverse equivalence is obtained by corestriction of the functor* $\mathcal{Y}_C$ *to the category* $\mathbf{S}_{\mathcal{Y}}\mathbf{F}$.

As an immediate consequence of 2.4 we notice:

2.11 LEMMA. *A join-preserving map* $c : C \to C'$ *is an* $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$*-morphism between* (S,C) *and* (S',C') *iff it induces an* $\mathbf{S}_{\mathcal{Y}}$*-morphism between* S *and* S'.

It follows from 2.8 that $\mathbf{S}_{\vec{\mathcal{Y}}}\mathbf{F}$ is a coreflective subcategory of $\mathbf{S}_{\vec{\mathcal{Y}}}\mathbf{C}$ and of $\mathbf{S}_{\vec{\mathcal{Y}}}\mathbf{E}$; similarly, from 2.9 we infer that $\mathbf{S}_{\mathcal{Y}}\mathbf{F}$ is a coreflective subcategory of $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$ and $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$. Moreover, we can prove:

2.12 THEOREM. $\mathbf{S}_{\vec{\mathcal{Y}}}\mathbf{C}$ *is a coreflective subcategory of* $\mathbf{S}_{\vec{\mathcal{Y}}}\mathbf{E}$, *and* $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$ *is a coreflective subcategory of* $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$.

PROOF. We only consider the second statement; the first one is treated similarly. Given an arbitrary $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$-object (S,C), we define

$$C_S = \{ \sup X : X \subseteq S \} \quad \text{(the joins formed in } C\text{)}.$$

C_S is closed under arbitrary joins in C and is a subsemigroup of C (because C is a complete residuated semigroup). Hence C_S is a complete residuated semigroup also, and the inclusion map $i_{C_S,C} : C_S \to C$ is a join-preserving semigroup embedding. On the other hand, the inclusion map $i_{S,C_S} : S \to C_S$ turns out to be a $\mathcal{Y}$-semigroup completion whence (S,C_S) is an object of the category $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$.
In order to show that $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$ is a coreflective subcategory of $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$ with coreflection maps

$$\varepsilon_{(S,C)} = i_{C_S,C} : (S,C_S) \hookrightarrow (S,C) ,$$

it remains to verify the following couniversal property: For each $\mathbf{S}_\gamma\mathbf{C}$-object (S,C) and each $\mathbf{S}_\gamma\mathbf{E}$-morphism $c : C \to C'$ between (S,C) and an $\mathbf{S}_\gamma\mathbf{E}$-object (S',C'), there exists a unique $\mathbf{S}_\gamma\mathbf{C}$-morphism $\tilde{c}$ between (S,C) and $(S',C'_{S'})$ such that $c = \varepsilon_{(S',C')} \circ \tilde{c}$. It is easy to see that the corestriction of c to $C'_{S'}$ (which is well-defined because c preserves joins and S is join-dense in C) is the only $\mathbf{S}_\gamma\mathbf{C}$-morphism $\tilde{c}$ with the required property.$_\square$

Summarizing some of the adjunction theorems derived before, we obtain the following diagram of adjoint functor pairs for an arbitrary isomorphism-closed standard completion γ.

$$\begin{tikzcd}
 & \mathbf{S}_\gamma\mathbf{E} \arrow[d, shift left, "\hookrightarrow"] \\
 & \mathbf{S}_\gamma\mathbf{C} \arrow[u, shift left, "\gamma_E"] \arrow[d, shift left, "\hookrightarrow"] \\
\mathbf{S}_\gamma \arrow[ur, shift left, "\gamma_C"] \arrow[r, shift left, "\simeq"] & \mathbf{S}_\gamma\mathbf{F} \arrow[u, shift left, "p_E"] \arrow[l, shift left, "p_C"]
\end{tikzcd}$$

$\hookrightarrow$ inclusion functor

$\downarrow$ coreflector

$\simeq$ equivalence

Now let us make the stronger assumption that γ is a compositive standard completion. Under this hypothesis the preceding definitions can be simplified a bit, and the adjunctions derived before are supplemented by a theorem characterizing γ as a reflector from the category $\mathbf{S}_\gamma$ to the subcategory $\mathbf{CS}$.

2.13 PROPOSITION. *Suppose γ is a compositive standard completion and S is a γ-semigroup. Then:*

(1) *The completion γS is a residuated γ-semigroup, and the embedding $\eta_S : S \to \gamma S$ is an $\mathbf{S}_\gamma$-morphism.*

(2) *If S' is a subsemigroup of S with $\downarrow y \cap S' \in \gamma S'$ for all $y \in S$ then S' is a γ-subsemigroup of S .*

(3) *Let (S,C) and (S',C') be $\mathbf{S}_\gamma\mathbf{C}$-objects. Then for a join-preserving map $c : C \to C'$, the following conditions are equivalent:*

(a) *c is an $\mathbf{S}_\gamma\mathbf{C}$-morphism between (S,C) and (S',C').*

(b) *c is a semigroup homomorphism between C and C' with $c[S] \subseteq S'$.*

(c) c <u>induces a semigroup homomorphism from</u> S <u>into</u> S'.

<u>PROOF</u>. (1) See 2.2.

(2) Let $x,z \in S'$ and $\downarrow z = \{ y \in S : y \leqq z \}$. If ℓ_x and ℓ'_x denote the left translations in S and in S', respectively, then $(\ell'_x)^{-1}[\downarrow z \cap S'] = \{ y \in S' : x \cdot y \leqq z \} = S' \cap \ell_x^{-1}[\downarrow z] \in \mathcal{Y}S'$ since ℓ_x and the inclusion map from S' into S are (weakly) $\mathcal{Y}$-continuous. Hence ℓ'_x is (weakly) $\mathcal{Y}$-continuous, too. Analogous reasoning holds for right translations.

(3) It is clear that (a) implies (b) and that (c) is a consequence of (b). In order to see that (c) implies (a), apply 2.11 and the fact that the restriction $c_0 : S \to S'$ is (weakly) $\mathcal{Y}$-continuous, because $c_0^{-1}[\downarrow y' \cap S'] = i_{S,C}{}^{-1}[c^{-1}[\downarrow y']]$ for $y' \in S'$. □

Now the announced reflection theorem (apply 2.5):

2.14 <u>THEOREM</u>. <u>Every compositive standard completion</u> $\mathcal{Y}$ <u>gives rise to a reflector from the category</u> $\mathbf{S}_{\mathcal{Y}}$ <u>to the subcategory</u> $\mathbf{CS}$. <u>Furthermore, the right projection functor</u> $\mathcal{Q}_E : \mathbf{S}_{\mathcal{Y}}\mathbf{E} \to \mathbf{CS}$ <u>with</u>

$\mathcal{Q}_E(S,C) = C$ <u>and</u> $\mathcal{Q}_E c = c$

<u>has a right adjoint right inverse, the diagonal embedding functor</u> $\mathcal{D}_E : \mathbf{CS} \to \mathbf{S}_{\mathcal{Y}}\mathbf{E}$ <u>with</u>

$\mathcal{D}_E C = (C,C)$ <u>and</u> $\mathcal{D}_E c = c$.

<u>In the following diagram of adjoint functor pairs the outer and the inner triangle commute</u>.

$$\begin{tikzcd}
 & \mathbf{S}_{\mathcal{Y}}\mathbf{E} \arrow[dl, shift right, "\mathcal{Y}_E"'] \arrow[dr, shift left, "\mathcal{Q}_E"] & \\
\mathbf{S}_{\mathcal{Y}} \arrow[ur, shift right, "\mathcal{P}_E"'] \arrow[rr, shift left, "\mathcal{Y}"] & & \mathbf{CS} \arrow[ul, shift left, "\mathcal{D}_E"] \arrow[ll, shift left, "\mathcal{Y}"]
\end{tikzcd}$$

<u>A similar triangle of adjoint functors is obtained for</u> $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$ <u>instead of</u> $\mathbf{S}_{\mathcal{Y}}\mathbf{E}$.

The functor $\mathcal{Y}_E$ induces an equivalence $\mathcal{Y}_F$ between $\mathbf{S}_{\mathcal{Y}}$ and a full coreflective subcategory $\mathbf{S}_{\mathcal{Y}}\mathbf{F}$ of $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$. On the other hand, the functor $\mathcal{D}_E$ induces an equivalence $\mathcal{D}_F$ between $\mathbf{CS}$ and a full reflective subcategory $\mathbf{FCS}$ of $\mathbf{S}_{\mathcal{Y}}\mathbf{C}$, and the following diagram of inclusion functors

commutes:

$$\begin{tikzcd}
 & \mathbf{S}_{\gamma}\mathbf{E} \arrow[d, dash] & \\
 & \mathbf{S}_{\gamma}\mathbf{C} & \\
\mathbf{S}_{\gamma}\mathbf{F} \arrow[ur, hook] \arrow[uur, hook] & & \mathbf{FCS} \arrow[ul, hook'] \arrow[uul, hook'] \\
\mathbf{S}_{\gamma} \arrow[u, "\gamma_F"', "\simeq"] \arrow[rr, hook] & & \mathbf{CS} \arrow[u, "\simeq"', "\mathcal{D}_F"]
\end{tikzcd}$$

($\hookrightarrow$ denotes coreflective subcategory, $\hookleftarrow$ denotes reflective subcategory.)

3. CONDITIONAL COMPLETIONS

In many contexts (in particular, when dealing with ordered algebraic systems such as partially ordered groups) it appears reasonable to consider standard extensions which are only conditionally complete and not necessarily complete (see e.g. [5],[15]). Before we can focus on the semigroup-theoretical applications, we need a few order-theoretical preliminaries.

A poset is called <u>conditionally complete</u> if each of its nonempty upper bounded subsets has a join, or equivalently, each nonempty lower bounded subset has a meet. A conditionally complete poset need not be a lattice: a nonempty conditionally complete poset is a join- (resp. meet-)semilattice iff it is up- (resp. down-)directed.

Every standard completion γ gives rise to a <u>conditional completion</u> γ^{o} where $\gamma^{o}P$ is defined for any poset P to be the collection of all nonempty upper bounded members of γP. Obviously, γ^{o} is a standard extension but not a standard completion. A straightforward computation shows that $\gamma^{o}P$ is always a conditionally complete poset with respect to inclusion, and $\gamma^{o}P$ is a join- (meet-)semilattice iff P is up- (down-)directed (or empty). All existing meets in $\gamma^{o}P$ agree with the corresponding intersections. If X is a nonempty upper bounded subset of P then so is $X^{-}=\bigcap\{ Y \in \gamma P : X \subseteq Y \}$. In particular, if X is a nonempty upper bounded subfamily of $\gamma^{o}P$ then $(\bigcup X)^{-}$ is not only the join of X in γP but also in $\gamma^{o}P$. Moreover, if $f: P \to P'$ is isotone (in particular, if f is weakly γ-continuous) and $X \subseteq P$ is

nonempty and upper bounded then so is $f[X]$, whence

$$f[X]^- = \bigcap\{Y' \in \mathcal{Y}P' : f[X] \subseteq Y'\} \in \mathcal{Y}^0P'.$$

Therefore, f can be lifted to a map

$$\mathcal{Y}^0f : \mathcal{Y}^0P \to \mathcal{Y}^0P', \quad Y \mapsto f[Y]^-.$$

Concerning the preservation of joins, we have:

3.1 LEMMA. *A weakly $\mathcal{Y}$-continuous map f between conditionally complete posets P and P' preserves nonempty joins iff it preserves $\mathcal{Y}^0$-joins*, i.e.,

$$f(\sup Y) = \sup f[Y] \quad \text{for } Y \in \mathcal{Y}^0P.$$

PROOF. The necessity of this condition is clear. Let X be a nonempty subset of P possessing a join x in P. Then $f(x)$ is an upper bound of $f[X]$ because f is isotone. If y is another upper bound of $f[X]$ then we have $X \subseteq f^{-1}[\downarrow y] \in \mathcal{Y}P$, and it follows that $X^- = \bigcap\{Y \in \mathcal{Y}^0P : X \subseteq Y\} \subseteq f^{-1}[\downarrow y]$. Now we have $f(x) = f(\sup X) = f(\sup X^-)$ and, if f preserves $\mathcal{Y}^0$-joins, then $f(\sup X^-) = \sup f[X^-] \leqq y$. □

In spite of this result, the transfer from completions to conditional completions has to be treated with care, as the following examples show.

3.2 EXAMPLES.

(1) A join-preserving map between conditionally complete lattices need not be weakly $\mathcal{Y}$-continuous. For example, the real-valued function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto x+y$ is join-preserving but not (weakly) $\mathcal{N}$-continuous since $f^{-1}[\downarrow 0] = \{(x,y) : x+y \leqq 0\}$ fails to be a cut in $\mathbb{R}\times\mathbb{R}$. In fact, $f^{-1}[\downarrow 0]^- = \mathbb{R}\times\mathbb{R}$.

(2) A map between complete lattices preserving nonempty joins need not be weakly $\mathcal{Y}$-continuous. For example, the embedding e of the singleton $\{1\}$ into the unit interval $[0,1]$ is a map between complete lattices which preserves nonempty joins, but e is not (weakly) $\mathcal{N}$-continuous since $e^{-1}[\{0\}] = \emptyset$ and $\emptyset$ is not an element of $\mathcal{N}\{1\}$.

(3) It may occur that a $\mathcal{Y}$-continuous map between complete lattices preserves nonempty joins but not the join of the empty set. In fact, the embedding in (2) is certainly isotone, i.e., $\mathcal{A}$-continuous, and preserves nonempty joins, but $e(\sup\emptyset) = e(1) = 1 \neq 0 = \sup\emptyset = \sup e[\emptyset]$.

(4) It may happen that a standard completion $\mathcal{Y}$ is union complete while the conditional completion $\mathcal{Y}^{\mathrm{o}}$ is not. For example, define $\mathcal{Y}$ as follows:

$$\mathcal{Y}P = \begin{cases} \mathcal{N}P \cup \{P \setminus \{1\}\} & \text{if } P \text{ is isomorphic to } Q = \\ \mathcal{A}P & \text{if } P \text{ is isomorphic to } \mathcal{Y}^{\mathrm{o}}Q \\ \mathcal{N}P & \text{otherwise.} \end{cases}$$

Then $\mathcal{Y}$ is union complete since $\mathcal{Y}\mathcal{Y}P = \mathcal{N}\mathcal{Y}P = \mathcal{M}\mathcal{Y}P$ for each poset P. But $\mathcal{Y}^{\mathrm{o}}$ is not union complete since $X = \{\{a\},\{b\}\} \in \mathcal{A}^{\mathrm{o}}\mathcal{Y}^{\mathrm{o}}Q = \mathcal{Y}^{\mathrm{o}}\mathcal{Y}^{\mathrm{o}}Q$ while $\bigcup X = \{a,b\}$ is not an element of $\mathcal{Y}^{\mathrm{o}}Q$.

In contrast to 3.2(4), we have the following useful compatibility property of standard completions.

3.3 LEMMA. *If $\mathcal{Y}$ is compositive then so is $\mathcal{Y}^{\mathrm{o}}$.*

PROOF. Suppose $f: P \to P'$ is weakly $\mathcal{Y}^{\mathrm{o}}$-continuous. Since $\mathcal{Y}^{\mathrm{o}}P$ is contained in $\mathcal{Y}P$ and $\mathcal{Y}$ is compositive, it follows that f is $\mathcal{Y}$-continuous. Thus $Y' \in \mathcal{Y}^{\mathrm{o}}P'$ implies $f^{-1}[Y'] \in \mathcal{Y}P$, and there exist elements $y' \in Y'$ and $z' \in P'$ with $Y \subseteq \downarrow z'$. Hence $f^{-1}[\downarrow y'] \subseteq f^{-1}[Y'] \subseteq f^{-1}[\downarrow z']$. As f is weakly $\mathcal{Y}^{\mathrm{o}}$-continuous, $f^{-1}[\downarrow y']$ is nonempty and $f^{-1}[\downarrow z']$ is upper bounded. Accordingly, $f^{-1}[Y']$ is nonempty and upper bounded, too.$_{\square}$

In order to extend our considerations from completions to conditional completions, we need a link between $\mathcal{Y}$ and $\mathcal{Y}^{\mathrm{o}}$. Thus we call a standard completion conditionable if for each poset P, the inclusion map from $\mathcal{Y}^{\mathrm{o}}P$ into $\mathcal{Y}P$ is weakly $\mathcal{Y}$-continuous. Explicitly stated, this condition means that $\mathcal{Y}^{\mathrm{o}}P$ is a $\mathcal{Y}$-subposet of $\mathcal{Y}P$, i.e.,

$$\{ X \in \mathcal{Y}^{\mathrm{o}}P : X \subseteq Y \} \in \mathcal{Y}\mathcal{Y}^{\mathrm{o}}P \quad \text{for all } Y \in \mathcal{Y}P.$$

Let us discuss two examples of compositive standard completions; one which is conditionable and another which is not.

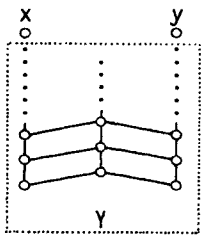
3.4 EXAMPLES.

(1) We know that the normal completion $\mathcal{N}$ is compositive (see Section 1). Hence by 3.3, the conditional normal completion $\mathcal{N}^{\mathrm{o}}$ is also compositive. Recall that the conditional completion of the rationals is a model of the real line, a pioneer result due to Dedekind. In order to see that $\mathcal{N}$ is conditionable, observe that

$$\mathcal{N}^{\mathrm{o}}P \cup \{P\} \subset \mathcal{N}P \subset \mathcal{N}^{\mathrm{o}}P \cup \{\emptyset,P\} .$$

Indeed, if $Y \in \mathcal{N}P$ is nonempty but not upper bounded then $Y = \{\downarrow x : Y \subseteq \downarrow x\} = \emptyset = P$. Hence, for any $Y \in \mathcal{N}P$ the system $\mathcal{X} = \{X \in \mathcal{N}^{o}P : X \subseteq Y\}$ is either empty or a principal ideal in $\mathcal{N}^{o}P$, or the whole poset $\mathcal{N}^{o}P$. But if $\mathcal{X}$ is empty then so is Y. In this case neither P nor $\mathcal{N}^{o}P$ can have a least element, and consequently $\mathcal{X} = \emptyset$ is a cut of $\mathcal{N}^{o}P$.

(2) The <u>Frink ideal completion</u> $\mathcal{F}^{\Delta}P$ of a poset P consists of all directed unions of cuts (see [1],[8],[9],[14]). It is the least algebraic closure system containing all principal ideals. We have already mentioned that $\mathcal{F}^{\Delta}$ (and so $\mathcal{F}^{\Delta o}$) is compositive. Now consider the following poset:



P

The Frink ideal Y is not upper bounded whence $Y \in \mathcal{F}^{\Delta}P \setminus \mathcal{F}^{\Delta o}P$. The system

$$\mathcal{X} = \{X \in \mathcal{F}^{\Delta o}P : X \subseteq Y\}$$

contains the ideals $\downarrow x \setminus \{x\}$ and $\downarrow y \setminus \{y\}$, and it is not a member of $\mathcal{F}^{\Delta}\mathcal{F}^{\Delta o}P$, because there is no bounded ideal containing these two ideals and so the only ideal of $\mathcal{F}^{\Delta o}P$ including $\mathcal{X}$ is the whole poset $\mathcal{F}^{\Delta o}P$. Thus $\mathcal{F}^{\Delta}$ cannot be conditionable.

Now let us study the canonical $\mathcal{Y}^{o}$-embeddings

$$\eta_P^{o} : P \to \mathcal{Y}^{o}P , \quad x \mapsto \downarrow x .$$

Part (2) of Proposition 1.1 translates without any problems to the setting of conditional completions.

3.5 <u>PROPOSITION. If $\mathcal{Y}$ is an arbitrary standard completion then for every $\mathcal{Y}$-continuous map $f : P \to P'$ there exists a unique map $h^{o} : \mathcal{Y}^{o}P \to \mathcal{Y}^{o}P'$ preserving nonempty joins and satisfying the equation $h^{o} \circ \eta_P^{o} = \eta_{P'}^{o} \circ f$. Moreover, h^{o} extends to a unique join-preserving map $h : \mathcal{Y}P \to \mathcal{Y}P'$ such that the following diagram commutes:</u>

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\eta^{\mathcal{O}}_P"] \arrow[dd, bend right=60, "\eta_P"'] & P' \arrow[d, "\eta^{\mathcal{O}}_{P'}"] \arrow[dd, bend left=60, "\eta_{P'}"] \\
\mathcal{Y}^{\mathcal{O}}P \arrow[r, dashed, "h^{\mathcal{O}}"] \arrow[d, hook, "i_P"] & \mathcal{Y}^{\mathcal{O}}P' \arrow[d, hook, "i_{P'}"] \\
\mathcal{Y}P \arrow[r, dashed, "h"] & \mathcal{Y}P'
\end{tikzcd}$$

The simple proof is left to the reader.

The translation of Part (1) in Proposition 1.1 to conditional completions is a bit more problematic than that of Part (2). In fact, it turns out that conditionability of $\mathcal{Y}$ is an indispensable tool in proving the desired universal property of the maps $\eta^{\mathcal{O}}_P$.

3.6 PROPOSITION. *The following statements are equivalent for an arbitrary standard completion $\mathcal{Y}$ and a poset P:*

(a) *The inclusion map $i_P : \mathcal{Y}^{\mathcal{O}}P \to \mathcal{Y}P$ is weakly $\mathcal{Y}$-continuous.*

(b) *For each weakly $\mathcal{Y}$-continuous map g from P into a (conditionally) complete poset C there exists a unique weakly $\mathcal{Y}$-continuous map $g^{\vee} : \mathcal{Y}^{\mathcal{O}}P \to C$ preserving nonempty joins with $g = g^{\vee} \circ \eta^{\mathcal{O}}_P$, viz. $g^{\vee}(Y) = \sup g[Y]$.*

(c) *For each $\mathcal{Y}$-continuous map f from P into an arbitrary poset P' there exists a unique weakly $\mathcal{Y}$-continuous map $\bar{f} : \mathcal{Y}^{\mathcal{O}}P \to \mathcal{Y}P$ preserving nonempty joins such that $\bar{f} \circ \eta^{\mathcal{O}}_P = \eta_{P'} \circ f$, viz. $\bar{f}(Y) = f[Y]^{-}$.*

Each of these conditions is fulfilled if $\mathcal{Y}$ is conditionable.

PROOF. (a) $\Rightarrow$ (b): Each set $Y \in \mathcal{Y}^{\mathcal{O}}P$ is nonempty and has an upper bound, and the same holds for the image $g[Y]$; hence $g^{\vee}(Y) = \sup g[Y]$ is well defined, and clearly $g = g^{\vee} \circ \eta^{\mathcal{O}}_P$. For $y \in C$, we obtain $g^{\vee -1}[\downarrow y] = \{ Y \in \mathcal{Y}^{\mathcal{O}}P : \sup g[Y] \leqq y \} = \{ Y \in \mathcal{Y}^{\mathcal{O}}P : Y \subseteq g^{-1}[\downarrow y] \} \in \mathcal{Y}\mathcal{Y}^{\mathcal{O}}P$ because $g^{-1}[\downarrow y]$ belongs to $\mathcal{Y}P$ and the inclusion map from $\mathcal{Y}^{\mathcal{O}}P$ into $\mathcal{Y}P$ is weakly $\mathcal{Y}$-continuous. Thus the map $g^{\vee}$ is weakly $\mathcal{Y}$-continuous, too. Now let $\mathcal{X}$ be a nonempty subset of $\mathcal{Y}^{\mathcal{O}}P$ possessing a join Y in $\mathcal{Y}^{\mathcal{O}}P$. Then Y is also the join of $\mathcal{X}$ in $\mathcal{Y}P$. As $g^{\vee}$ is isotone, $g^{\vee}(Y)$ is an upper bound of $g^{\vee}[\mathcal{X}]$. If y is any other upper bound of $g^{\vee}[\mathcal{X}]$ in C then we have $g^{\vee}(X) \leqq y$, i.e. $X \subseteq g^{-1}[\downarrow y] \in \mathcal{Y}P$ for all $X \in \mathcal{X}$,

and it follows that $Y = \sup X \subseteq g^{-1}[\downarrow y]$, i.e., $g^{\vee}(Y) \leq y$. This shows that $g^{\vee}(Y) = \sup g^{\vee}[X]$, and consequently $g^{\vee}$ preserves nonempty joins. By this condition and the equation $g = g^{\vee} \circ \eta_P^0$ the map $g^{\vee}$ is uniquely determined because η_P^0 is join-dense. But notice that $g^{\vee}$ need not preserve the empty join (and, in particular, $g^{\vee}$ is not always residuated). For example, the constant map $g: \{0,1\} \to \{0,1\}$, $x \mapsto 1$ is certainly isotone, i.e. A-continuous, but it is not A^0-continuous, and $g^{\vee}$ does not preserve all joins because $g^{\vee}(\sup \emptyset) = g^{\vee}(\{0\}) = g(0) = 1 \neq 0 = \sup \emptyset = \sup g^{\vee}[\emptyset]$.

(b) $\Rightarrow$ (c): If $f: P \to P'$ is $\mathcal{Y}$-continuous then the map $g = \eta_{P'} \circ f$ from P to $\mathcal{Y}P'$ is weakly $\mathcal{Y}$-continuous (cf.[9,2.4]), so (b) applies to this map. But $g^{\vee}(Y) = \sup \eta_{P'}[f[Y]] = \sup\{\downarrow f(y) : y \in Y\} = f[Y]^{-} \in \mathcal{Y}^0P' \subseteq \mathcal{Y}P'$ for $Y \in \mathcal{Y}^0P$ (joins formed in $\mathcal{Y}P'$).

(c) $\Rightarrow$ (a): Take for f the identity on P. Then $\overline{f}(Y) = f[Y]^{-} = Y^{-} = Y$ for all $Y \in \mathcal{Y}^0P$, i.e., $\overline{f}$ is the inclusion map from $\mathcal{Y}^0P$ into $\mathcal{Y}P$. $\square$

<u>REMARK</u>. The assumption of conditionability can be avoided if we restrict our attention to weakly $\mathcal{Y}^0$-continuous maps instead of weakly $\mathcal{Y}$-continuous maps. In fact, the same arguments as in 1.1 show that a map g from a poset P into a conditionally complete poset C is weakly $\mathcal{Y}^0$-continuous iff there exists a (unique) residuated map $g: \mathcal{Y}^0P \to C$ with $g = g^{\vee} \circ \eta_P^0$. However, for most applications it is more convenient to work with weakly $\mathcal{Y}$-continuous maps if $\mathcal{Y}$ is known to be conditionable.

Now consider the following subcategory $\mathbf{C}_{\mathcal{Y}}^0$ of $\mathbf{P}_{\mathcal{Y}}$ (see Section 1). The objects of $\mathbf{C}_{\mathcal{Y}}^0$ are all conditionally complete posets, and the morphisms are those $\mathcal{Y}$-continuous maps between them which preserve nonempty joins. As an immediate consequence of Propositions 3.5 and 3.6, we obtain:

3.7 <u>THEOREM</u>. <u>If $\mathcal{Y}$ is a compositive and conditionable standard completion then $\mathbf{C}_{\mathcal{Y}}^0$ is a reflective subcategory of $\mathbf{P}_{\mathcal{Y}}$ with reflector $\mathcal{Y}^0$ and reflection maps η_P^0.</u>

On the morphism level, the functor $\mathcal{Y}^0$ assigns to each $\mathcal{Y}$-continuous map $f: P \to P'$ the unique $\mathbf{C}_{\mathcal{Y}}^0$-morphism $h^0: \mathcal{Y}^0P \to \mathcal{Y}^0P'$ described in 3.5 (that this map is actually $\mathcal{Y}$-continuous for compositive and conditionable $\mathcal{Y}$ follows from 3.6).

A large collection of compositive and conditionable standard completions is obtained from subset systems. Recall that we have associated with every subset system Z the join-ideal completion $Z^{\vee}$ where

$$Z^{\vee}P = \{ Y \in AP : \text{if } Z \in ZP, Z \subseteq Y \text{ and } x = \sup Z \text{ then } x \in Y \}$$

and that $Z^{\vee}$ is always compositive. Now we can prove:

3.8 <u>PROPOSITION</u>. <u>For every subset system Z the join-ideal completion $Z^{\vee}$ is conditionable.</u>

<u>PROOF</u>. Put $\mathcal{Y} = Z^{\vee}$. We have to show that for any $Y \in \mathcal{Y}P$ the set $\mathcal{Y}_0 = \{ X \in \mathcal{Y}^0P : X \subseteq Y \}$ belongs to $\mathcal{Y}\mathcal{Y}^0P$, in other words, that whenever a set $\mathcal{X} \in Z\mathcal{Y}^0P$ is a subset of $\mathcal{Y}_0$ and has a join Y_0 in $\mathcal{Y}^0P$ then $Y_0 \in \mathcal{Y}_0$. We know that $Y_0 = (\bigcup \mathcal{X})^-$ provided that $\mathcal{X}$ is nonempty. On the other hand, if $\mathcal{X}$ is empty then we have $\emptyset \in Z\mathcal{Y}^0P$ and consequently $\emptyset \in ZP$, since any constant map $f : \mathcal{Y}^0P \to P$ transports the empty set from $Z\mathcal{Y}^0P$ to ZP (the case $P = \emptyset$ is trivial). The join of $\mathcal{X} = \emptyset$ in $\mathcal{Y}^0P$ could differ from that in $\mathcal{Y}P$ only if P had a least element 0 and $\emptyset \in \mathcal{Y}P = Z^{\vee}P$. But this is impossible because $\emptyset \in ZP$ and $\emptyset \in Z^{\vee}P$ would imply $0 = \sup \emptyset \in \emptyset$. Accordingly, the equation $Y_0 = (\bigcup \mathcal{X})^-$ holds for $\mathcal{X} = \emptyset$ as well. Now the inclusion $\mathcal{X} \subseteq \mathcal{Y}_0$ implies $\bigcup \mathcal{X} \subseteq Y$, and as Y belongs to $\mathcal{Y}P$, we obtain $Y_0 = (\bigcup \mathcal{X})^- \subseteq Y$, i.e., $Y_0 \in \mathcal{Y}_0$. □

From this result, we infer that the standard completions $\mathcal{A} = \mathcal{M}^{\vee}$, $\mathcal{J} = \mathcal{A}^{\vee}$, $\mathcal{F}^{\vee}$, and $\mathcal{D}^{\vee}$ (see Section 1) are conditionable, while the Frink ideal completion $\mathcal{F}^{\Delta}$ is not, by 3.4(2). This example shows that $Z^{\vee}$ cannot be replaced with the Δ-ideal completion Z^{Δ} in 3.8.

In order to formulate Theorem 3.7 for the present situation of join-ideal completions, let us denote by $\mathbf{C}^0$ the following category: The objects of $\mathbf{C}^0$ are the conditionally complete posets, and the morphisms are maps between them which preserve nonempty joins. $\mathbf{C}^0_+$ will denote that subcategory of $\mathbf{C}^0$ which has the same objects but only those morphisms which preserve arbitrary joins - including the join of the empty set (provided this join exists). Then 3.7 in connection with 3.8 leads to the following:

3.9 <u>COROLLARY</u>. <u>Let Z be any subset system. If $\emptyset$ is not an element of ZP for one (hence every) nonempty poset P then the category</u>

$\mathbf{C}^o$ is a reflective subcategory of $\mathbf{P}_{Z^\vee}$, the category of posets and Z-join preserving maps. If, on the other hand, $\emptyset \in ZP$ for one (hence every) nonempty poset P then the category $\mathbf{C}^o_+$ is a reflective subcategory of $\mathbf{P}_{Z^\vee}$.
In both cases the reflector is $Z^{\vee o}$, and the reflection maps are the embeddings

$$\eta^o_P : P \to Z^{\vee o}P , \quad x \mapsto \downarrow x .$$

In the non-compositive case, suitable modifications of the definitions and results in [9] and [11] provide adjunction theorems, similar to 2.8 and 2.9, for conditional completions. We only discuss one special result. Let Y be an isomorphism-closed and conditionable standard completion. As objects of the category $\mathbf{P}_Y\mathbf{C}^o$ we take those pairs (P,C) where C is a conditionally complete poset and P is a subposet of C such that the inclusion map from P into C is weakly Y-continuous and join-dense, i.e.

$$\downarrow y \cap P \in YP \quad \text{and} \quad y = \sup(\downarrow y \cap P) \quad \text{for all} \quad y \in C .$$

A morphism between two $\mathbf{P}_Y\mathbf{C}^o$-objects (P,C) and (P',C') is a map c between C and C' that preserves nonempty joins and induces a Y-continuous map c_o from P into P'. Thus we have a forgetful projection functor $\mathcal{P}^o_C : \mathbf{P}_Y\mathbf{C}^o \to \mathbf{P}_Y$ with

$\mathcal{P}^o_C(P,C) = P$ for $\mathbf{P}_Y\mathbf{C}^o$-objects (P,C) and
$\mathcal{P}^o_C c = c_o$ for $\mathbf{P}_Y\mathbf{C}^o$-morphisms c.

In the other direction, we have a "conditional completion functor" $Y^o_C : \mathbf{P}_Y \to \mathbf{P}_Y\mathbf{C}^o$ with

$Y^o_C P = (MP, Y^o P)$ for posets P and
$Y^o_C f = Y^o f$ for Y-continuous maps f.

Now it is a straightforward exercise to derive the following adjunction theorem from 3.6:

3.10 THEOREM. Suppose Y is an isomorphism-closed and conditionable standard completion. Then the conditional completion functor Y^o_C is left adjoint to the left projection functor $\mathcal{P}^o_C$, and the unit of the adjunction is an isomorphism. If, in addition, Y is compositive then the diagonal embedding functor $\mathcal{D}^o_C : \mathbf{C}^o_Y \to \mathbf{P}_Y\mathbf{C}^o$ with $\mathcal{D}^o_C C = (C,C)$ and $\mathcal{D}^o_C c = c$ is right adjoint to the right projection functor $\mathcal{Q}^o_C : \mathbf{P}_Y\mathbf{C}^o \to \mathbf{C}^o_Y$ with $\mathcal{Q}^o_C(P,C) = C$ and

$\mathcal{Q}_C^o c = c$. <u>The following diagram of adjoint functors commutes:</u>

$$\begin{tikzcd}
 & \mathbf{P}_{\mathcal{Y}}\mathbf{C}^o \arrow[dl, shift right, "\mathcal{Y}_C^o"'] \arrow[dr, shift left, "\mathcal{Q}_C^o"] & \\
\mathbf{P}_{\mathcal{Y}} \arrow[ur, shift right, "\mathcal{P}_C^o"'] \arrow[rr, shift right, "\mathcal{Y}^o"'] & & \mathbf{C}_{\mathcal{Y}}^o \arrow[ul, shift left, "\mathcal{D}_C^o"] \arrow[ll, shift right]
\end{tikzcd}$$

This theorem applies, for example, to every join-ideal completion $\mathcal{Y} = Z^{\vee}$ where Z is an arbitrary subset system, but also to the normal completion $\mathcal{N}$ by cuts.

Now given an arbitrary $\mathcal{Y}$-semigroup S, we may associate with S not only the completion $\mathcal{Y}S$ but also the conditional completion $\mathcal{Y}^oS$. But, in contrast to the situation with $\mathcal{Y}S$, we cannot hope that $\mathcal{Y}^oS$ always becomes a residuated semigroup (see 2.2). For example, any conditionally complete chain C (such as the real line) is an $\mathcal{N}$-semigroup with respect to the binary meet operation. But if C has no greatest element then the conditional completion $\mathcal{N}^oC = \mathcal{M}C$ is isomorphic to C and cannot be residuated. In fact, for $x \leqq y$ in C the inverse image $\ell_x^{-1}[\downarrow y] = \{ z \in C : x \wedge z \leqq y \} = C$ is never a principal ideal. However, we can prove (cf. 2.1(3)):

3.11 <u>PROPOSITION</u>. <u>For every $\mathcal{Y}$-semigroup S the conditional completion $\mathcal{Y}^oS$ is a lower semicontinuous subsemigroup of $\mathcal{Y}S$. The natural embedding</u>

$$\eta_S^o : S \to \mathcal{Y}^oS \ , \quad x \mapsto \downarrow x$$

<u>is a weakly $\mathcal{Y}$-continuous semigroup homomorphism. If S is residuated then so is $\mathcal{Y}^oS$, and η_S^o preserves the residuation operations.</u>

<u>PROOF</u>. Recall that a po-semigroup S is lower semicontinuous iff $x \cdot \sup Z = \sup(x \cdot Z)$ and $\sup Z \cdot x = \sup(Z \cdot x)$ holds for all nonempty subsets Z of S. This is certainly fulfilled if S is a subsemigroup of a complete residuated semigroup C such that $\sup_C Z \in S$ for all nonempty subsets Z of S. In particular, we infer from 2.2 that $\mathcal{Y}^oS$ is lower semicontinuous, being a subsemigroup of $\mathcal{Y}S$ and closed under nonempty joins in $\mathcal{Y}S$. Clearly, $\eta_S^o : S \to \mathcal{Y}^oS$ is weakly $\mathcal{Y}$-continuous

since $\eta_S : S \to \mathcal{Y}S$ is. Now suppose S is residuated. If $Z \in \mathcal{Y}^oS$ and u is an upper bound for Z then $u \cdot\!\!\cdot x$ is an upper bound for $\ell_x^{-1}[Z]$. Hence for each $X \in \mathcal{Y}^oS$, the set $Z \cdot\!\!\cdot X = \bigcap\{\ell_x^{-1}[Z] : x \in X\}$ (cf. 2.2) is upper bounded (observe that $X \neq \emptyset$). Now choose any element $z \in Z$ and any upper bound y of X. Then $z \cdot\!\!\cdot y$ is an element of $Z \cdot\!\!\cdot X$; in fact, for each $x \in X$ we have $x \leqq y$ and therefore $z \cdot\!\!\cdot y \leqq z \cdot\!\!\cdot x$, whence $x \cdot (z \cdot\!\!\cdot y) \in \downarrow z \subseteq Z$, i.e., $z \cdot\!\!\cdot y \in \ell_x^{-1}[Z]$. Thus the set $Z \cdot\!\!\cdot X$ is a nonempty upper bounded member of $\mathcal{Y}S$. This and a similar argument for right translations shows that $\mathcal{Y}^oS$ is a subalgebra of the residuated semigroup $\mathcal{Y}S$, with respect to the residuation operations.$_\square$

In general $\mathcal{Y}^oS$ will not be a $\mathcal{J}$-semigroup, i.e., the join of the empty set need not be preserved by all translations. For example, if we consider any conditionally complete but not complete lattice C with least element 0, then C is certainly an $\mathcal{A}$-semigroup with respect to the binary join operation, and consequently $\mathcal{A}C$ is a complete residuated semigroup, whereas $\mathcal{A}^oC$ fails to be a $\mathcal{J}$-semigroup since

$$C \vee \sup\emptyset = C \vee \{0\} = \{x \vee 0 : x \in C\} = C \neq \{0\} = \sup\emptyset .$$

However, if S is a $\mathcal{Y}$-semigroup without least element, or else a $\mathcal{Y}$-semigroup with least element 0 satisfying $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$, then it is easy to see that $\mathcal{Y}^oS$ is a $\mathcal{J}$-semigroup.

It should be mentioned that for a <u>po-group</u> G (see 2.1(7)) the conditional completion $\mathcal{Y}^oG$ rather rarely happens to become a po-group, too. By a theorem of Iwasawa and Ogasawara (cf.[15,V.12]), $\mathcal{Y}^oG$ (and therefore G) must be Abelian if $\mathcal{Y}^oG$ is a po-group. For the conditional completion by cuts $\mathcal{N}^o$ one knows that $\mathcal{N}^oG$ is a po-group if and only if G is <u>integrally closed</u>, i.e., $x^n \leqq y$ for all natural numbers n implies $x \leqq e$, the neutral element of G (cf.[15,V.17]).

As an immediate consequence of 2.13(2), we have

3.12 <u>COROLLARY</u>. <u>If $\mathcal{Y}$ is a compositive and conditionable standard completion then for each $\mathcal{Y}$-semigroup S the conditional completion $\mathcal{Y}^oS$ is a $\mathcal{Y}$-subsemigroup of the complete $\mathcal{Y}$-semigroup $\mathcal{Y}S$.</u>

The categories introduced in Section 2 admit obvious modifications for conditional completions. Objects of the category $\mathbf{C}^o\mathbf{S}$ are the conditionally complete lower semicontinuous semigroups (where translations

preserve nonempty joins). A morphism between two $\mathbf{C}^0\mathbf{S}$-objects C and C' is a semigroup homomorphism from C into C' preserving nonempty joins. The subcategory $\mathbf{C}^0_Y\mathbf{S} = \mathbf{S}_Y \cap \mathbf{C}^0\mathbf{S}$ is obtained in the obvious manner. Objects are conditionally complete lower semicontinuous Y-semigroups, and morphisms between them are Y-continuous semigroup homomorphisms preserving nonempty joins. On the other hand, the category $\mathbf{S}_Y\mathbf{C}^0$ is defined as follows: As objects take those pairs (S,C) where C is a $\mathbf{C}^0\mathbf{S}$-object and S is a join-dense Y-subsemigroup of C. A morphism between two such objects (S,C) and (S',C') is a semigroup homomorphism $c: C \to C'$ that preserves nonempty joins and induces a Y-continuous map between S and S'. Lemma 2.4 provides another characterization of $\mathbf{S}_Y\mathbf{C}^0$-morphisms.

3.13 COROLLARY. *Let (S,C) and (S',C') be $\mathbf{S}_Y\mathbf{C}^0$-objects. Then a join-preserving map $c: C \to C'$ is a morphism in $\mathbf{S}_Y\mathbf{C}^0$ iff it induces an $\mathbf{S}_Y$-morphism between S and S'.*

The semigroup modification of 3.6 reads as follows (cf. 2.5):

3.14 PROPOSITION. *If Y is conditionable then for each weakly Y-continuous semigroup homomorphism g from a Y-semigroup S into a $\mathbf{C}^0\mathbf{S}$-object C there is a unique weakly Y-continuous semigroup homomorphism $g^\vee: Y^0S \to C$ preserving nonempty joins and satisfying the equation $g = g^\vee \circ \eta^0_S$.*

Using this universal property of the embeddings η^0_S, we obtain:

3.15 THEOREM. *Let Y be an isomorphism-closed and conditionable standard completion. Then the conditional completion functor $Y^0_{\mathbf{C}} : \mathbf{S}_Y \to \mathbf{S}_Y\mathbf{C}^0$ which maps objects S to (MS, Y^0S) and morphisms f to Y^0f is left adjoint to the forgetful projection functor $P^0_{\mathbf{C}} : \mathbf{S}_Y\mathbf{C}^0 \to \mathbf{S}_Y$. If, in addition, Y is compositive then the category $\mathbf{C}^0_Y\mathbf{S} = \mathbf{S}_Y \cap \mathbf{C}^0\mathbf{S}$ is a reflective subcategory of $\mathbf{S}_Y$, with reflector Y^0 and reflection maps η^0_S.*

3.16 COROLLARY. *Let Z be any subset system such that $\emptyset \notin ZS$. Then the category $\mathbf{C}^0\mathbf{S}$ is a reflective subcategory of $\mathbf{S}_{Z^\vee}$, the category of $Z^\vee$-semigroups and Z-join preserving semigroup homomorphisms. Similarly, if $\emptyset \notin ZS$ then the category $\mathbf{C}^0_+\mathbf{S}$ (where morphisms preserve arbitrary joins) is a reflective subcategory of $\mathbf{S}_{Z^\vee}$.*

Suitable restrictions and corestrictions of the involved functors produce completion theorems for categories of po-groups etc.

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