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On solving the p-th complex auxiliary equation $f^{(p)}(z) = z$

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In this article we solve the first and second real auxiliary exponential equations using Lambert's W function. We exhibit a class of functions which extend W. We solve analytically the first, second and p-th complex auxiliary exponential equations using these functions, and give an analytic characterization of the domains of periodic points of order $p > 1$ for the complex iterated exponential $z^{z^{\cdots^z}}$. We then analytically solve transcendental equations with iterated exponential terms using a similar class of functions, and finally derive exact expressions for the derivatives and integrals of all functions involved.

Keywords: Auxiliary equation; Lambert's W function; Infinite exponential; Transcendental equation; Entire function; Fixed point; Periodic point

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Introduction

When $f(z) = c^z$, De Villiers and Robinson in [1, p. 13] introduce the terminology *n-th auxiliary equation* for the equation $f^{(n)}(z) = z$, with $f^{(n)}$ denoting the n-fold iterate of f. In particular, $f(z) = z$ is referred to as the *first auxiliary equation* and $f^{(2)}(z) = z$ is referred to as the *second auxiliary equation*, when dealing with iterated exponentials. We are interested in determining when the iterated exponential $z^{z^{\cdots^z}}$ goes into a stable p-cycle. For this, we are interested in finding the fixed points of period $p \geq 1$ of the function $f(z)$. This reduces to finding the (non-trivial) solutions of the p-th complex auxiliary equation, $f^{(p)}(z) = z$. Although there exists an analytic expression via Lambert's W function, which provides for all the solutions of $f(z) = z$ (which are the fixed points of period 1 of $f(z)$, and thus trivial solutions of $f^{(p)}(z) = z$), as $W(k, -\log(c))/(-\log(c)), k \in \mathbb{Z}$ († see Lemma 5.1), there is no analytic expression

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(using elementary functions) that provides for non-trivial solutions of $f^{(p)}(z) = z$, if $p \geq 2$, even for the real cases. Here we begin by analysing the first and second real auxiliary equations $f(x) = x$ and $f^{(2)}(x) = x$, using W and then, using suitably defined functions which extend W, we proceed to generalize (†), in the direction of complex numbers, thereby exhibiting an expression, which provides for fixed points of period $p > 1$ of $f(z)$. We remark that the domains of periodic points of period $p \geq 2$ are only known by computational methods, when imaging the parameter map of iterates of $f(z)$, but otherwise they do not have a characterization in terms of elementary functions or even W, since complicated complex dynamics are involved. We finally exhibit a class of related functions which can analytically solve transcendental equations containing terms with iterated exponentials.

1. Notation and preliminary lemmas

For the terminology about *periodic points* of functions, we refer the reader to Bergweiler [2, pp. 2–3] and Branner [3, pp. 37–41].

For the terminology *analytic function* we use the one found in Churchill and Brown in [4, p. 46].

Whenever a complex function is multi-valued and a parameter $k \in \mathbb{Z}$ is required to indicate the branch chosen, the omission of k altogether indicates always the principal branch of this function ($k = 0$). In particular, equations with complex exponents throughout this article are always understood to use the principal branch of complex exponentiation, whenever necessary: $c^w = e^{w \log(c)}$, $c \neq 0$, with $\log(z)$ being the principal branch of the complex function $\log(k, z)$, $k \in \mathbb{Z}$. The real counterparts of all functions in this article will be denoted as having real arguments x instead of z to avoid any confusion.

Using the principal branch of the complex log function, we use Maurer's notation for successive power iterates and the infinite iterate (see Knoebel [5, pp. 239–240]).

Definition 1.1 For $z \in \mathbb{C} \setminus \{x \in \mathbb{R}: x \leq 0\}$ and $n \in \mathbb{N}$,

$$ {}^{n}z = \begin{cases} z & \text{if } n = 1, \\ z^{({}^{n-1}z)} & \text{if } n > 1. \end{cases} $$

Definition 1.2 Whenever the following limit exists and is finite,

$$ {}^{\infty}z = \lim_{n \to \infty} {}^{n}z $$

We also use the exponential function $f(z)$ and its iterates. For $c \in \mathbb{C} \setminus \{0, 1\}$,

$$ f(z) = c^z \tag{1.1} $$

$$ f^{(n)}(z) = \begin{cases} f(z) & \text{if } n = 1, \\ f(f^{(n-1)}(z)) & \text{if } n > 1. \end{cases} \tag{1.2} $$

${}^{n}z$ and $f^{(n)}(z)$ are related: ${}^{n}c = f^{(n)}(1)$. Equations with real arguments will be denoted as having arguments x, to avoid any confusion. For example,

Definition 1.3 $g(x) = f(x) - x$.

Definition 1.4 $h(x) = f^{(2)}(x) - x$.

For $c > 0$, continuity for $f(x), g(x)$ and $h(x)$ follows from the definitions and elementary analysis, so it is implicitly assumed throughout sections 3 and 4. f, g and h are analytic everywhere and derivatives of all orders exist everywhere.

We also use Lambert's W function. For the essential properties of this function we refer the reader to Knuth, Corless and Jeffrey [6, pp. 344–349], [7, pp. 199–205], and for a summary of its properties in the complex plane to [8, p. 762]. Briefly, it is defined as the function that satisfies,

$$W(z)e^{W(z)} = z, \quad z \in \mathbb{C} \tag{1.3}$$

or the function which solves for z the equation,

$$ze^z = a, \quad a, z \in \mathbb{C}. \tag{1.4}$$

W is multi-valued and as such is denoted as $W(k, z)$ with k specifying the branch chosen. $W(z)$ always denotes the principal branch ($k=0$) and the corresponding real function will be denoted as $W(x)$. Only the two branches of W corresponding to $k=0$ and $k=-1$ can ever assume real values. In particular, we will use the following lemmas (see [8, p. 762] for 1.5–1.9 and [6, pp. 331] for 1.10).

Lemma 1.5 $W(k, z) \in \mathbb{R} \Rightarrow k \in \{-1, 0\}$.

Lemma 1.6 *$W(x)$ is real valued, continuous and strictly increasing on the interval $[-e^{-1}, +\infty)$ and is complex valued outside this interval.*

Lemma 1.7 *$W(-1, x)$ is real valued, continuous and strictly decreasing on the interval $[-e^{-1}, 0)$ and is complex valued (or undefined at 0) outside this interval.*

Lemma 1.8 $W(-1, -e^{-1}) = W(-e^{-1}) = -1$.

Lemma 1.9 $W(e) = 1$ *and* $W(0) = 0$.

Lemma 1.10 *If* $x \in [-e^{-1}, 0)$ *then* $W(-1, x) \leq W(x)$.

Lemma 1.11 *If* $0 < x < e^{-e}$ *then* $W(-\ln(x)) > -W(1/\ln(x))$.

Proof Note that $\ln(x) < -e \Rightarrow -\ln(x) > e \Rightarrow W(-\ln(x)) > 1$ and $1/\ln(c) > -e^{-1} \Rightarrow W(1/\ln(x)) > -1 \Rightarrow -W(1/\ln(x)) < 1$ by Lemmas 1.6 and 1.9, consequently $W(-\ln(x)) > -W(1/\ln(x))$ and the lemma follows. ■

Lemma 1.12 *If* $0 < x < e^{-e}$ *then* $W(-\ln(x)) < -W(-1, 1/\ln(x))$.

Proof Let $j(x) = W(-\ln(x)) + W(-1, 1/\ln(x))$ for $0 < x < e^{-e}$. Corless *et al.* in [7, p. 201] give the derivative of W as $\mathrm{d}W(k,x)/\mathrm{d}x = W(k,x)/(x(1+W(k,x)))$, for $k \in \mathbb{Z}$, therefore $\mathrm{d}j/\mathrm{d}x = 0 \Rightarrow -(W(-1,1/\ln(x)) - W(-\ln(x)))/((1+W(-1,1/\ln(x)))x\ln(x)\times(1+W(-\ln(x)))) = 0 \Rightarrow W(-1,1/\ln(x)) = W(-\ln(x))$. Remembering W's Definition (1.3) and applying xe^x to the last equation, we get $\ln(x)^2 = -1 \Rightarrow x \in \{e^{-i}, e^i\} \Rightarrow x \in \mathbb{C}$, so $j(x)$ has no critical points, therefore it is either increasing or decreasing throughout the range $0 < x < e^{-e}$. Now $j(e^{-e}) = 0$, by Lemmas 1.8 and 1.9 and numerical evaluation shows that $\mathrm{d}j/\mathrm{d}x > 0$, for any $0 < x < e^{-e}$, consequently $j(x)$ is increasing throughout this range, reaching a maximum at $x = e^{-e}$. But then, necessarily $j(x) < 0$ throughout this range, and the lemma follows. ■

In view of W being the 'inverse' of ze^z, there is nothing that prevents one from considering 'inverses' of more general functions. In fact, it seems that Knuth *et al.* may have considered such functions, but after an extensive search, the author has not become aware of any systematic expositions (perhaps because there is quite a large variety), relating to the functions similar to the ones that will be presented herein.

Let $c_i \in \mathbb{C} \setminus \{0\}$. Define $F_{n,m}(z)$: $\mathbb{N}^2 \times \mathbb{C} \to \mathbb{C}$ as follows.

Definition 1.13

$$F_{n,m}(z) = \begin{cases} e^z & \text{if } n = 1, \\ e^{c_{m-(n-1)} F_{n-1,m}(z)} & \text{if } n > 1. \end{cases}$$

Definition 1.14 $G(c_1, c_2, \ldots, c_k; z) = zF_{k+1,\, k+1}(z)$.

If $k=0$, then $G(z) = ze^z$. If $k=1$ then $G(c_1; z) = ze^{c_1 e^z}$. If $k=2$ then $G(c_1, c_2; z) = ze^{c_1 e^{c_2 e^z}}$. When we write about G in general, we may use $G(\ldots; z)$, omitting the variable list of parameters. The function that will ultimately interest us in the subsequent sections of this article, is the 'inverse' of $G(\ldots; z)$, if and whenever it makes sense. Momentarily disregarding issues of existence (whether such an object exists altogether), we introduce this new 'object', called (the) (possibly multi-valued) 'inverse' of G, by:

$$HW(c_1, c_2, \ldots, c_k; y). \tag{1.5}$$

In other words, this 'object' should satisfy,

$$G(\ldots; HW(\ldots; y)) = y \tag{1.6}$$

assuming that the list of arguments for both G and HW is identical. These 'functions' will henceforth be called HW (according to 'Hyper W') so as to not deviate from the standard notation for Lambert's W function. Using this notation, note that when $k=0$, $HW(y) = W(y)$ is Lambert's W function. Note also that (1.6) is a more general form of the equation (1.3), which W satisfies.

2. Existence theorems and calculations of inverses

The existence of HW in all cases is guaranteed by the Lagrange Inversion Theorem, which we repeat here for completeness (see Saks and Sygmund [9, pp. 201–202]).

THEOREM 2.1 *If a function $G(z)$ is holomorphic in the neighbourhood of a point $z_0 \neq \infty$, and if $G'(z_0) \neq 0$, then it is uniquely invertible in a certain neighbourhood of the point z_0. Its inverse function $H(w)$ is holomorphic in the neighbourhood of the point $w_0 = G(z_0)$, and is therefore expansible in a neighbourhood of this point in a power series with center w_0, as*

$$H(w) = z_0 + \sum_{n=1}^{\infty} \frac{(w - w_0)^n}{n!} \left\{ \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[\frac{z - z_0}{G(z) - w_0} \right]^n \right\}_{z=z_0}. \tag{2.1}$$

Equation (2.1), can be used for example to calculate a series expansion for Lambert's W function about the origin, as $W(z) = \sum_{n=1}^{\infty} (-n)^{n-1} z^n / n!$ (see Knuth *et al.* [6, pp. 344–349], [7, pp. 199–205] or [8, p. 763]).

Theorem 2.1 allows for calculating the inverse about the origin (note that for all G in Definition 1.4, $\{dG(\ldots;z)/dz\}_{z=0} \neq 0$), but it is not very useful for calculations outside the region of convergence, particularly if this region is small. Here we choose to use Newton's method, instead, for the actual implementation.

The idea now, is that if we have an analytic G and the equation $G(\ldots;z) = y$, perhaps if we can numerically solve the equation $T_n(z) = y$ (for a certain finite n), where $T_n(z)$ is the Taylor polynomial of degree n of G, we may be able to obtain an approximate zero. Once we have such an approximation, we can give this approximation to Newton's method to get better estimates of the zero. In the Appendix, we give the actual algorithm in Maple. Here we prove that this algorithm will work.

Lemma 2.2 *Given* $c_i \in \mathbb{C}$, $i \in \{1, 2, \ldots, k\}$, $y \in \mathbb{C}$, $n \in \mathbb{N}$, *the algorithm in the Appendix, finds an* $r \in \mathbb{C}$, *such that* $G(c_1, c_2, \ldots, c_k; r) = y$.

Proof Fix n and let $u(z) = G(\ldots;z) = y$. Let $v(z) = u(z) - y$, and let $v_n(z) = T_n(z) - y$, where $T_n(z)$ is the Taylor polynomial of degree n in the expansion of $u(z)$. $v_n(z) \to v(z)$ uniformly and all functions are entire. In particular $v(z)$ is entire and non-constant, so for all y (with at most one exception) there exists r such that $u(r) = y$ by Picard's Little Theorem, therefore there exists r (except in at most one case) such that $v(r) = 0$. Once we have a root r, Hurwitz's Root Theorem (see section 5.3.4 of [10, p. 76]) provides for a neighbourhood $|z - r| < \delta$, and a number m, such that within the neighbourhood $|z - r| < \delta$, for all $n > m$, $v_n(z) = 0$ has a root in that neighbourhood. In particular, there exists r_n in $|r_n - r| < \delta$, such that $T_n(r_n) = y$. Now $|y - v(r_n)| < \epsilon$, using the continuity of $v(z)$, and the code picks the r_n that minimizes ϵ. Note that if $v(r) = 0$ and $v_n(r_n) = 0$ then $\lim_{n\to\infty} \inf |r_n - r| = 0$, by the uniform convergence of $v_n(z)$, so increasing the degree of the Taylor polynomials, n, gives us better approximations r_n of the root r, which can be given to Newton's algorithm for improvement in accuracy. This completes the part of the proof which provides for a good approximate zero. For the rest of the proof, (i.e. convergence of Newton's method) the area has been investigated thoroughly, so we refer the reader to Barnsley [11, pp. 280–285] and Peitgen and Richter [12, pp. 93–102] for an introduction and for deeper results to Friedman [13, p. 6] (Theorem 2.2), Ocken [14, p. 241] (Theorem 1), Haruta [15, p. 2505] (Theorem 2) and Wang and Zhao [16, p. 259] (Theorems 1.1, 1.2 and 2.4), where necessary and sufficient conditions for convergence are given. ■

We note that the first part of the algorithm essentially suffices in finding a root even without giving the approximate zero to Newton's method, by increasing n sufficiently, since $\lim_{n\to\infty} \inf |r_n - r| = 0$. On the other hand, Haruta's Theorem 2 guarantees that by increasing n we can force r_n to fall inside the basin of attraction of a root r, since the basin of attraction always has at least finite area. Of particular interest here are the theorems by Smale and Wang and Han (Theorems 1.1 and 1.2) which provide for an Alpha-test for the approximate zeros. Wang's Theorem 2.4 guarantees that there exists a neighbourhood around the root r, such that if $|r - z| < \delta$, then we have convergence for Newton's method, which is essentially how we use the algorithm. We finally note that the algorithm converges even at the branch points, although no longer quadratically (see Conte and De Boor [17, pp. 103–109] and Friedmann [13, p. 5]). A simple modification to the algorithm with a suitable multiplier m can be used to force quadratic convergence even at those points.

It should be noted here that the HW functions in general are multi-valued and possess many branches. For example $HW(-\log(c); y)$ has a main branch point, which can be found by solving the equation $dG/dz = 0$ using W's Definition (1.3). $dG(-\log(c); z)/dz = 0 \Rightarrow -e^{-\log(c)e^z}(-1 + z\log(c)e^z) = 0 \Rightarrow ze^z = 1/\log(c) \Rightarrow z = W(1/\log(c))$, thus $HW(-\log(c); y)$ must have a branch point at $z_0 = G(-\log(c); W(1/\log(c))) = W(1/\log(c))e^{c/(\log(c)W(1/\ln(c)))}$.

Similar calculations based on Lemmas 1.6 and 1.8, show that if $0 < c \leq e^{-e}$ then $G(-\log(c); W(1/\log(c))) \in \mathbb{R}$ and furthermore, by taking the limit as $c \to 0^+$ and using the definition of W (1.4), the branch point z_0 lies on the negative real axis, and satisfies: $-1.02457 \doteq -e^{e^{-e-1}} \leq z_0 < 0$, consequently for such c the main branch cut of the principal branch of $HW(-\log(c); y)$ is $BC_{HW} = (-\infty, z_0)$.

For the more general functions $HW(c; y)$, solving $dG/dz = 0$ gives $z = W(-1/c)$, and the main branch point of HW lies at $z_0 = G(c; W(-1/c))$. If $c \geq e$ then $G(c; W(-1/c)) \in \mathbb{R}$ and furthermore, the main branch point $z_0 = G(c; W(-1/c)) = W(-1/c)e^{-1/W(-1/c)}$ lies on the negative real axis, and satisfies: $-\infty < z_0 \leq -e$, consequently for such c the main branch cut of the principal branch of $HW(c; y)$ is $BC_{HW} = (-\infty, z_0)$.

Because W is itself multi-valued, the $HW(c; y)$ functions really have infinitely many branch points $z_k = G(c; W(k, -1/c)) = W(k, -1/c)e^{-1/W(k,-1/c)}$, $k \in \mathbb{Z}$. The branch cuts can be either taken to be rays extending from the branch points to ∞ in any direction, or can be taken as a single curve which just joins all the branch points together. The precise identification of all the branch cuts for $HW(c; y)$ is beyond the scope of this article, as it is dependent on the particular algorithm one uses. We present here an example of what $HW(c; y)$ looks like plotted using the magnitude and argument of the function, for $c = 0.02 + 0.02i$, in figure 1. The function's closest branch points are at $z_0 \doteq 1.59077 + 1.66087i$ and at $z_{-1} \doteq 1.32741 - 2.90302i$. The algorithm in the Appendix generates the appropriate branch cuts for the principal branch, extending from these branch points in the direction of $-\infty$. For more information on identifying branch cuts on Riemann surfaces for multi-valued functions, we refer the reader to Jeffrey [18, pp. 3–8] where the author uses some non-trivial examples.

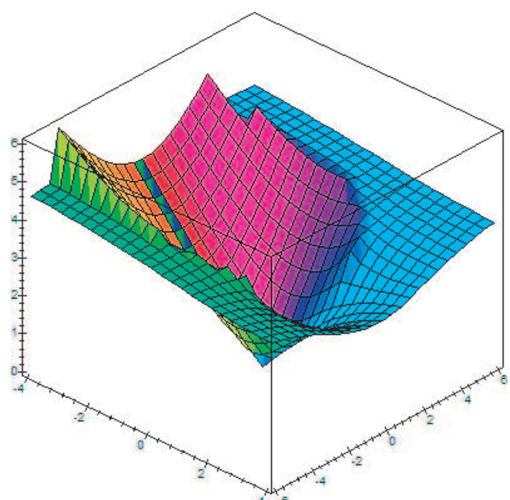

Figure 1. $HW(c; y)$, $c = 0.02 + 0.02i$.

For the general functions $HW(\ldots;y)$ the situation is much more complex and there may be no way to even identify all the branch points, since the equation $\mathrm{d}G(\ldots;z)/\mathrm{d}z = 0$ may not be analytically solvable in general. In this case, numerical methods must be applied.

Theorem 2.1 guarantees that all functions $HW(\ldots;y)$ are analytic at 0, with a Lagrange series expansion $\sum_{n=1}^{\infty} a_n y^n$ whose coefficients are given by equation (2.1) as

$$a_n = \frac{1}{n!}\left\{\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[\frac{1}{F_{k+1,\,k+1}(z)}\right]\right\}_{z=0}.$$

The radius of convergence for these expansions will be the distance to the closest branch point, $R = \inf\{|G(\ldots;z)|\colon \mathrm{d}G(\ldots;z)/\mathrm{d}z = 0\}$. (For example for the function in figure 1, the radius of convergence will be $R = |z_0| \doteq 2.3$.)

Note that Theorem 2.1 guarantees that $HW(\ldots;y)$ is continuous in any neighbourhood of $y_0 = G(\ldots;z_0)$ provided $[\mathrm{d}G(\ldots;z)/\mathrm{d}z]_{z=z_0} \neq 0$. At the branch cuts, continuity is preserved if we use the notion of counterclockwise continuity around the branch points when passing to other branches. Since we will be using only the principal branch of HW, these functions are necessarily discontinuous at their branch cuts (see figure 1).

3. Solving the first auxiliary real exponential equation $f(x) = x$ using Lambert's W function

In this section $c > 0$ and $x \in \mathbb{R}$ throughout.

Lemma 3.1 *If* $c > e^{e^{-1}}$ *then* $g(x) = 0$ *admits no real roots.*

Proof $g(x) = 0$ is analytically solvable via Lambert's W function. $g(x) = 0 \Rightarrow f(x) = x \Rightarrow xe^{-x\ln(c)} = 1 \Rightarrow -x\ln(c)e^{-x\ln(c)} = -\ln(c) \Rightarrow -x\ln(c) = W(k, -\ln(c)) \Rightarrow x = W(k, -\ln(c))/(-\ln(c)) = e^{-W(k,-\ln(c))}, k \in \mathbb{Z}$, using W's definition (1.4). If $c > e^{e^{-1}}$ then $-\ln(c) < -e^{-1}$ consequently if $k \in \{-1, 0\}$ then $W(k, -\ln(c)) \in \mathbb{C}$ by Lemmas 1.6 and 1.7 and therefore $x \in \mathbb{C}$. ■

Lemma 3.2 *If* $c = e^{e^{-1}}$ *then* $g(x) = 0$ *admits exactly one real root,* $x_0 = e$.

Proof As in Lemma 3.1, and note that $-\ln(c) = -e^{-1}$, consequently if $k \in \{-1, 0\}$ the values of $W(-1, -\ln(c))$ and $W(-\ln(c))$ coincide by Lemma 1.8 and therefore $x_{-1} = x_0 = e$. ■

Lemma 3.3 *If* $1 < c < e^{e^{-1}}$ *then* $g(x) = 0$ *admits exactly two real roots* $x_k = e^{-W(k,-\ln(c))}$, $k \in \{-1, 0\}$.

Proof As in Lemma 3.1 and note that $-e^{-1} < -\ln(c) < 0$, consequently if $k \in \{-1, 0\}$, then $W(k, -\ln(c)) \in \mathbb{R}$, by Lemmas 1.6 and 1.7 and therefore $x_k \in \mathbb{R}$. ■

Lemma 3.4 *If* $e^{-e} < c < 1$ *then* $g(x) = 0$ *admits exactly one real root,* $x_0 = e^{-W(-\ln(c))}$.

Proof As in Lemma 3.1 and note that $0 < -\ln(c) < e$, thus $W(-\ln(c)) \in \mathbb{R}$ by Lemma 1.6, and $W(-1, -\ln(c)) \in \mathbb{C}$ by Lemma 1.7 and therefore $x_0 \in \mathbb{R}$ and $x_{-1} \in \mathbb{C}$ and the lemma follows. ■

LEMMA 3.5 *If* $c = e^{-e}$ *then* $g(x) = 0$ *admits exactly one real root*, $x_0 = e^{-1}$.

Proof As in Lemma 3.1 and note that $-\ln(c) = e$, consequently $W(-\ln(c)) \in \mathbb{R}$ by Lemma 1.6 and $W(-1, -\ln(c)) \in \mathbb{C}$ by Lemma 1.7 and therefore $x_0 \in \mathbb{R}$ and $x_{-1} \in \mathbb{C}$ and the lemma follows. ∎

LEMMA 3.6 *If* $0 < c < e^{-e}$ *then* $g(x) = 0$ *admits exactly one real root*, $x_0 = e^{-W(-\ln(c))}$.

Proof As in Lemma 3.1 and note that $-\ln(c) > e$, consequently $W(-\ln(c)) \in \mathbb{R}$ by Lemma 1.6 and $W(-1, -\ln(c)) \in \mathbb{C}$ by Lemma 1.7 and therefore $x_0 \in \mathbb{R}$ and $x_{-1} \in \mathbb{C}$ and the lemma follows. ∎

4. Solving the second auxiliary real exponential equation $f^{(2)}(x) = x$ using Lambert's W function

We will need the following lemmas:

LEMMA 4.1 *If* $c > 1$ *then* $\mathrm{d}h/\mathrm{d}x = 0$ *admits exactly one real root*, x_{crit}, *with* $x_{\text{crit}} = \ln(W(1/\ln(c))/\ln(c))/\ln(c)$.

Proof $\mathrm{d}h/\mathrm{d}x = 0$ is analytically solvable via Lambert's W function. $\mathrm{d}h/\mathrm{d}x = 0 \Rightarrow f^{(2)}(x)f(x)\ln(c)^2 = 1 \Rightarrow e^{\ln(c)f(x)}\ln(c)f(x) = 1/\ln(c) \Rightarrow \ln(c)f(x) = W(k, 1/\ln(c))$ by the definition of W (1.4), thus $f(x) = W(k, 1/\ln(c))/\ln(c)$, thus $x_{\text{crit},k} = \ln(W(k, 1/\ln(c))/\ln(c))/\ln(c)$, $k \in \mathbb{Z}$. Verify that $c > 1 \Rightarrow 1/\ln(c) > 0 \Rightarrow W(1/\ln(c)) \in \mathbb{R}$ by Lemma 1.6, and $W(-1, 1/\ln(c)) \in \mathbb{C}$ by Lemma 1.7, consequently $x_{\text{crit},0} \in \mathbb{R}$ and $x_{\text{crit},-1} \in \mathbb{C}$ and the lemma follows. ∎

LEMMA 4.2 *If* $e^{-e} < c < 1$ *then* $\mathrm{d}h/\mathrm{d}x = 0$ *admits no real roots.*

Proof As in Lemma 4.1 and note that $1/\ln(c) < -e^{-1}$, consequently if $k \in \{-1, 0\}$ then $W(k, 1/\ln(c)) \in \mathbb{C}$ and therefore $x_{\text{crit},k} \in \mathbb{C}$. ∎

LEMMA 4.3 *If* $c = e^{-e}$ *then* $\mathrm{d}h/\mathrm{d}x = 0$ *admits exactly one real root*, $x_{\text{crit}} = e^{-1}$.

Proof As Lemma 4.1 and note that $1/\ln(c) = -e^{-1}$, consequently if $k \in \{-1, 0\}$ the values of $W(-1, 1/\ln(c))$ and $W(1/\ln(c))$ coincide by Lemma 1.8 and therefore $x_{\text{crit},-1} = x_{\text{crit},0} = e^{-1}$. ∎

LEMMA 4.4 *If* $0 < c < e^{-e}$ *then* $\mathrm{d}h/\mathrm{d}x = 0$ *admits exactly two real roots*, $x_{\text{crit},k}$, *with* $x_{\text{crit},k} = \ln(W(k, 1/\ln(c))/\ln(c))/\ln(c)$, $k \in \{-1, 0\}$ *and* $x_{\text{crit},-1} < x_{\text{crit},0}$.

Proof As in Lemma 4.1 and note that $-e^{-1} < 1/\ln(c) < 0$, consequently if $k \in \{-1, 0\}$, then $W(k, 1/\ln(c)) \in \mathbb{R}$, by Lemmas 1.6 and 1.7 and therefore $x_{\text{crit},k} \in \mathbb{R}$. The final inequality follows from Lemma 1.10. ∎

We are now ready for the main lemmas of this section.

LEMMA 4.5 *If* $c > e^{e^{-1}}$ *then* $h(x) = 0$ *admits no real roots.*

Proof For x_{crit} of Lemma 4.1, it suffices to show $h(x_{\text{crit}}) > 0$ and that h possesses a minimum there. After simplifying and expanding logarithms and using W's Definition (1.3), we find that $h(x_{\text{crit}}) = 1/(\ln(c)W(1/\ln(c))) + 2\ln(\ln(c))/\ln(c) + W(1/\ln(c))/\ln(c)$. Using the substitution $\ln(c) = (ae^a)^{-1}$, for $a = W(1/\ln(c))$, using W's Definition (1.4) for $x = 1/\ln(c)$, and simplifying the terms remembering that

$W(ae^a) = a$, we finally obtain $h(x_{\text{crit}}) = -e^a(-1 + 2a\ln(a) + a^2) = m(a)$, for $a = W(1/\ln(c))$. Note that $0 < 1/\ln(c) < e$, therefore $W(0) < W(1/\ln(c)) < W(e)$ and thus $0 < a < 1$ by Lemmas 1.6 and 1.9. Elementary calculus shows that $m(a) > 0$ for $0 < a < 1$. On the other hand, $\mathrm{d}^2h/\mathrm{d}x^2|_{x_{\text{crit}}} = \ln(c)[W(1/\ln(c)) + 1] > 0$, by Lemmas 1.6 and 1.8 and the lemma follows. ■

Lemma 4.6 *If* $c = e^{e^{-1}}$ *then* $h(x) = 0$ *admits exactly one real root* $x_0 = e$.

Proof As in Lemma 4.5 and note that $x_{\text{crit}} = e = x_0$, $h(x_{\text{crit}}) = 0$, h possesses a minimum there since $\mathrm{d}^2h/\mathrm{d}x^2|_{x_{\text{crit}}} = 2/e > 0$ and the lemma follows. ■

Lemma 4.7 *If* $1 < c < e^{e^{-1}}$ *then* $h(x) = 0$ *admits exactly two real roots* $x_k = e^{-W(k,\,-\ln(c))}$, $k \in \{-1, 0\}$.

Proof As in Lemma 4.5, note that $1 < c < e^{e^{-1}}$, and thus $a > 1$ by Lemmas 1.6 and 1.9, consequently $m(a) < 0$ and again $\mathrm{d}^2h/\mathrm{d}x^2|_{x_{\text{crit}}} > 0$, by Lemmas 1.6 and 1.8. Note that $-e^{-1} < -\ln(c) < 0$, consequently if $k \in \{-1, 0\}$ then $W(k,\,-\ln(c)) \in \mathbb{R}$ by Lemmas 1.6 and 1.7, thus $x_k \in \mathbb{R}$ and verify that x_k are roots. $f^{(2)}(x_k) = c^{e^{-W(k,\,-\ln(c))}} = e^{-W(k,\,-\ln(c))} = x_k$, using the definition of W, (1.3), so $h(x_k) = 0$ and the lemma follows. ■

Lemma 4.8 *If* $e^{-e} < c < 1$ *then* $h(x) = 0$ *admits exactly one real root* $x_0 = e^{-W(-\ln(c))}$.

Proof h has no critical points by Lemma 4.2. Note $e^{-e} < c < 1 \Rightarrow 0 < -\ln(c) < e$, so if $y_{\text{crit}} = \ln(-1/\ln(c))/\ln(c)$, then $y_{\text{crit}} \in \mathbb{R}$ and y_{crit} is a critical point of the function $\mathrm{d}h/\mathrm{d}x$. Furthermore, $\mathrm{d}^3h/\mathrm{d}x^3|_{y_{\text{crit}}} = \ln(c)^3/e < 0$, so $\mathrm{d}h/\mathrm{d}x$ possesses a maximum there. On the other hand, $\mathrm{d}h/\mathrm{d}x|_{y_{\text{crit}}} = -e^{-1}\ln(c) - 1 < 0$, so $\mathrm{d}h/\mathrm{d}x < 0$ throughout. Therefore h has at most one real root. We verify as in Lemma 4.7 that $-W(-\ln(c)) \in \mathbb{R}$ by Lemma 1.6 and therefore $x_0 \in \mathbb{R}$, so h has at least one real root and the lemma follows. ■

Lemma 4.9 *If* $c = e^{-e}$ *then* $h(x) = 0$ *admits exactly one real root* $x_0 = e^{-1}$.

Proof As in Lemma 4.8, if $x \in \mathbb{R} \setminus \{e^{-1}\}$ then $\mathrm{d}h/\mathrm{d}x < 0$. Verify that $x_{\text{crit}} = e^{-1}$ of Lemma 3.3 satisfies $h(x_{\text{crit}}) = 0$ and the lemma follows. ■

Lemma 4.10 *If* $0 < c < e^{-e}$ *then* $h(x) = 0$ *admits exactly three real roots* $\{x_1, x_0, x_2\}$ *with* $x_0 = e^{-W(-\ln(c))}$ *and* $x_1 < x_0 < x_2$.

Proof This case is depicted in figure 2. h has exactly two critical points by Lemma 4.4. $\mathrm{d}^2h/\mathrm{d}x^2|_{x_{\text{crit},0}} = \ln(c)[W(1/\ln(c)) + 1] < 0$ and $\mathrm{d}^2h/\mathrm{d}x^2|_{x_{\text{crit},-1}} = \ln(c)[W(-1, 1/\ln(c)) + 1] > 0$, by Lemmas 1.6, 1.7 and 1.8, so h possesses a local maximum at $x_{\text{crit},0}$ and a local minimum at $x_{\text{crit},-1}$. Next verify that $h(x) = 0$ admits the real root x_0 as in Lemma 4.7. We now show that $x_{\text{crit},-1} < x_0 < x_{\text{crit},0}$. Algebraic calculations using the definition of W (1.3), show that the inequality $x_0 < x_{\text{crit},0}$ is equivalent to Lemma 1.11, while $x_{\text{crit},-1} < x_0$ is equivalent to Lemma 1.12, both of which have been shown in section 1. Next we show that $h(x_{\text{crit},-1}) < 0$ and $h(x_{\text{crit},0}) > 0$. For the first inequality suppose $h(x_{\text{crit},-1}) \geq 0$. If $h(x_{\text{crit},-1}) > 0$, then by the continuity of h, since $x_{\text{crit},-1}$ is a local minimum, and since $h(x_0) = 0$, then there exists a critical point in $(x_{\text{crit},-1}, x_0)$ which is a local maximum, a contradiction, since the only critical points of $\mathrm{d}h/\mathrm{d}x$ are given by Lemma 4.4 and $x_{\text{crit},0} > x_0$. Therefore $h(x_{\text{crit},-1}) \leq 0$. If $h(x_{\text{crit},-1}) = 0$, again since $h(x_0) = 0$, by the continuity of h either there exists a critical point in $(x_{\text{crit},-1}, x_0)$, which is either a local minimum or local maximum or $h(x) = 0$ identically in $[x_{\text{crit},-1}, x_0]$, both of which are contradictions, so we finally conclude $h(x_{\text{crit},-1}) < 0$.

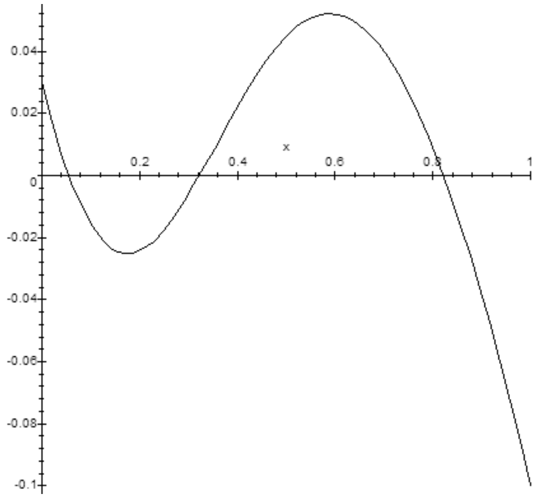

Figure 2. $h(x)$, $c = 0.03$, $0 \leq x \leq 1$.

The inequality $h(x_{\text{crit},0}) > 0$ is proved similarly. Next note that $h(0) = c > 0$, so by the Intermediate Value Theorem, $h(x) = 0$ admits at least one real root $x_1 \in (0, x_{\text{crit},-1})$. If $h(x) = 0$ admitted more than one real root in $(0, x_{\text{crit},-1})$, then continuity would again imply the existence of at least one critical point in $(0, x_{\text{crit},-1})$, a contradiction. Therefore $h(x) = 0$ admits exactly one real root there. Similarly, $h(1) = c^c - 1 < 0$, so again by the Intermediate Value Theorem, $h(x) = 0$ admits exactly one real root $x_2 \in (x_{\text{crit},0}, 1)$, using similar reasoning. The ordering of the roots follows immediately from the inequalities $x_1 < x_{\text{crit},-1} < x_0 < x_{\text{crit},0} < x_2$ and Lemma 4.4 and the lemma follows. ■

Similar results to Lemmas 4.5–4.10 are shown in De Villiers and Robinson [1, pp. 15–21] without using Lambert's W function, using standard analysis.

In [8, p. 763] and Corless *et al.* [6, p. 332] it is shown that whenever ${}^{\infty}c$ exists finitely in 1.2, it equals $h(c) = W(-\log(c))/(-\log(c)) = e^{-W(-\log(c))}$, while Ash [19, pp. 207–208] and Macdonnell [20, pp. 301–303] establish that for $k \in \mathbb{N}$, $\lim_{c \to 0^+} {}^{2k}c = 1$ and $\lim_{c \to 0^+} {}^{2k+1}c = 0$ and whenever $c \in (0, e^{-e})$, $\{{}^n c\}_{n \in \mathbb{N}}$ is a 2-cycle, by considering the even and odd subsequences, ${}^{2n}c$ and ${}^{2n+1}c$. The bifurcation which occurs and its behaviour and properties are analysed in Ash [19, p. 207], De Villiers and Robinson [1, p. 15] and Macdonnell [20, p. 299]. The two branches stemming from the bifurcation point $\{e^{-e}, e^{-1}\}$ can be parametrized as $a^{a/(1-a)}$ and $a^{1/(1-a)}$ for appropriate positive a. (see for example Knoebel [5, p. 237] or Voles [21, p. 212].) In this case as shown in Spivak [22, p. 434], Knoebel [5, pp. 241–243], De Villiers and Robinson [1, p. 13] and Lense [23, p. 501], the two separate limits $a = \lim_{n \to \infty} {}^{2n+1}c$ and $b = \lim_{n \to \infty} {}^{2n}c$ satisfy $0 < a < h(c) < b < 1$ and the *second auxiliary equation system*,

$$\begin{Bmatrix} a = c^{c^a} \\ b = c^a \end{Bmatrix}. \tag{4.1}$$

Note that $x_0 = h(c)$ and x_0 is a repeller. If $0 < c < e^{-e}$ then $-\ln(c) > e$, consequently $W(-\ln(c)) > 1$, by Lemmas 1.6 and 1.9, therefore $\mathrm{d}(f^{(2)})/\mathrm{d}x|_{x_0} = f^{(2)}(x_0)f(x_0)\ln(c)^2 = x_0^2\ln(c)^2 = W(-\ln(c))^2 > 1$. Also $f^{(2)}(b) = f^{(3)}(a) = f(a) = b$ by system 4.1, so $\{a, b\} = \{x_1, x_2\}$ by Lemma 4.10, although we do not know yet the order of $\{x_1, x_2\}$. We will derive analytic expressions for $\{x_1, x_2\}$ in section 6.

5. Solving the first auxiliary complex exponential equation $f(z) = z$ using Lambert's W function

The first auxiliary complex equation $f(z) = z$ has an analytic solution via Lambert's W function. $z = f(z) \Leftrightarrow z = c^z \Leftrightarrow ze^{-z\log(c)} = 1 \Leftrightarrow -z\log(c)e^{-z\log(c)} = -\log(c) \Leftrightarrow -z\log(c) = W(k, -\log(c))$, $k \in \mathbb{Z}$, by Lambert's Definition 1.3, so $z = W(k, -\log(c))/(-\log(c))$, $k \in \mathbb{Z}$, or alternatively, again using W's definition, $z = e^{-W(k, -\log(c))}$, $k \in \mathbb{Z}$. In effect, W gives all the fixed points of f in the complex plane. In [8, p. 764] we show that the only potential attractor and thus the only potential limit of the Infinite Exponential ${}^{\infty}c$ is given by the principal branch of W.

Lemma 5.1 *Whenever* ${}^{\infty}c$ *exists finitely, its value is given by*:

$$h(c) = \frac{W(-\log(c))}{-\log(c)}.$$

6. Solving the second auxiliary complex exponential equation $f^{(2)}(z) = z$ using an HW function

The general solution of Lemma 5.1 concerns, of course, the case of $c \in \mathbb{C}$ where $\{{}^n c\}_{n\in\mathbb{N}}$ converges. (See Shell [24, p. 679], [25, p. 12], Baker and Rippon [26, p. 106], [27, p. 502], [28] and [8, pp. 768–772] for details.) Numerical evidence has shown that there are many regions on the complex plane, where the iteration $\{{}^n c\}_{n\in\mathbb{N}}$ is a p-cycle, with p distinct subsequences converging to separate limits. (See for example Baker and Rippon [27, p. 503].)

In those cases, i.e. when the iteration is a p-cycle, the distinct limits of the convergent subsequences are given as $\{z, f(z), \ldots, f^{(p-1)}(z)\}$, where z is a fixed point of the p-th auxiliary equation $f^{(p)}(z) = z$, but it is not a fixed point of any n-th auxiliary equation $f^{(n)}(z) = z$, with $n < p$. If we therefore solve this equation non-trivially, we can hope to classify the domains of periodic convergence in the complex plane.

We are ready for the main lemma of this section.

Lemma 6.1 *If* $c \in \mathbb{C}\backslash\{0, 1\}$, *then the second auxiliary complex exponential equation* $f^{(2)}(z) = z$, *admits the solutions* $\{z, f(z)\}$, *where*

$$z = \frac{HW(-\log(c); \log(c))}{\log(c)}.$$

Proof The equation $f^{(2)}(z) = z$ is analytically solvable by an appropriate HW function, in a similar way as in section 5. $f^{(2)}(z) = z \Leftrightarrow c^{c^z} = z \Leftrightarrow zc^{-c^z} = 1 \Leftrightarrow ze^{-\log(c)e^{z\log(c)}} = 1 \Leftrightarrow z\log(c)e^{-\log(c)e^{z\log(c)}} = \log(c) \Leftrightarrow we^{-\log(c)e^{w}} = \log(c)$, after setting

$w = z\log(c)$. The last equation admits a solution via HW as $w = HW(-\log(c); \log(c)) \Leftrightarrow z = HW(-\log(c); \log(c))/\log(c)$. If z is a solution, then $f^{(2)}(z) = z$, therefore $f^{(2)}(f(z)) = f(f^{(2)}(z)) = f(z)$, so $f(z)$ is also a solution and the lemma follows. ■

Note that in this case the corresponding HW function that provides the solution is the inverse of $G(-\log(c); z) = ze^{-\log(c)e^z}$, $c \in \mathbb{C}$. Therefore immediately,

Corollary 6.2 *Whenever $c \neq 0$ and $\{^n c\}_{n\in\mathbb{N}}$ is an attracting 2-cycle, the limits of the two subsequences $\{^{n+k} c\}_{n\in\mathbb{N}}$, $k \in \{0, 1\}$ are given as $\{z, f(z)\}$, where z is given by Lemma 6.1.*

We note the similarity of the expression for the fixed points of $f^{(2)}(z)$ with that for the fixed points of $f(z)$ in section 5.

The multiplier of the fixed point z of $f^{(n)}(z)$ is given in Shell [25, p. 28] as,

$$\left(f^{(n)}\right)'(z) = \log(c)^n \prod_{k=1}^{n} f^{(k)}(z). \tag{6.1}$$

Example In section 4 we mentioned that if $0 < c < e^{-e}$ then $\{^n c\}_{n\in\mathbb{N}}$ is an attracting 2-cycle. We now have a means to calculate the fixed points and thus use them to check the fixed point condition. For $n = 2$, equation (6.1) reduces to $(f^{(2)})'(z) = f^{(2)}(z)f(z)\log(c)^2$. Therefore if $c = 0.03$ for example, then $z \doteq 0.05613$ by Lemma 6.1, and a numerical calculation shows $|f^{(2)}(z)f(z)\log(c)^2| \doteq 0.56688 < 1$, therefore $\{^n c\}_{n\in\mathbb{N}}$ is an attracting 2-cycle and furthermore, $\lim_{n\to\infty} {}^{2n+1}c \doteq 0.05613$ by Lemma 6.1, and consequently the two limits of the 2-cycle are 0.05613 and $c^{0.05613} \doteq 0.82132$ by Corollary 6.2. These are exactly the first and third real roots $\{x_1, x_2\}$ of Lemma 4.10, which are shown in figure 2.

One last observation remains before we can make a conjecture about all 2-cycles. Let D be the unit disk and consider the map $\phi: \mathbb{C} \mapsto \mathbb{C}$, defined as: $\phi(z) = e^{z/e^z}$. The image of D under ϕ is a certain nephroid region $N = \phi(D)$. This region is shown in figure 3. Shell in [24, p. 679], [25, p. 12] and Baker and Rippon in [26, p. 106] show that if c belongs to the interior of N, then $\{^n c\}_{n\in\mathbb{N}}$ converges. Furthermore, Baker and Rippon in [27, p. 502] and [28] establish Theorem 6.3.

Theorem 6.3 (Baker and Rippon) *$\{^n c\}_{n\in\mathbb{N}}$ converges for $\log(c) \in \{te^{-t}: |t| < 1$, or $t^n = 1$, for some $n \in \mathbb{N}\}$ and it diverges elsewhere.*

In [8, p. 768] we show that $\phi^{-1}(c) = -W(-\log(c))$ and that therefore Theorem 6.3 is equivalent to Theorem 6.4.

Theorem 6.4 *$\{^n c\}_{n\in\mathbb{N}}$ converges if $|\phi^{-1}(c)| < 1$ or $(\phi^{-1}(c))^n = 1$, for some $n \in \mathbb{N}$ and it diverges elsewhere.*

Redefine $N = \{c \in \mathbb{C}: |\phi^{-1}(c)| < 1 \text{ or } (\phi^{-1}(c))^n = 1\}$. Whenever $\{^n c\}_{n\in\mathbb{N}}$ converges, it converges to $W(-\log(c))/(-\log(c))$ given in Lemma 5.1. Now note that if $c \in N$ then $\{^n c\}_{n\in\mathbb{N}}$ is a 1-cycle (since it converges), so in this case the values given in Lemmas 5.1 and 6.1 must *a fortiori* coincide (see Corollary 7.4), therefore $HW(-\log(c); \log(c))/\log(c) = W(-\log(c))/(-\log(c))$. With this information we can now use equation (6.1) to formulate a conjecture about all $c \in \mathbb{C}$, such that $\{^n c\}_{n\in\mathbb{N}}$ is a 2-cycle.

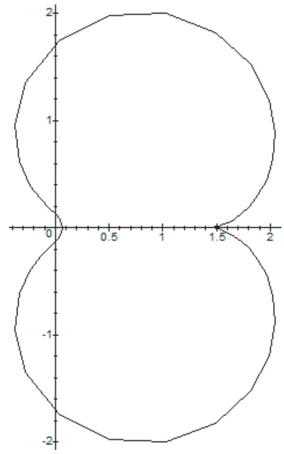

Figure 3. $N = \phi(D)$.

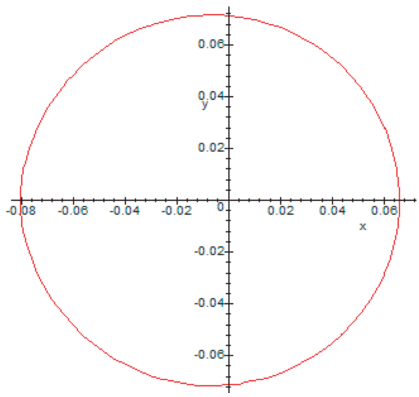

Figure 4. $\{c \in \mathbb{C}: |te^t| = |\log(c)^{-1}|\}$.

LEMMA 6.5 *If* $c \in \mathbb{C} \backslash N$ *and* $t = HW(-\log(c); \log(c))$, *then* $\{^n c\}_{n \in \mathbb{N}}$ *is an attracting 2-cycle if* $|te^t| < |\log(c)^{-1}|$ *and can be so only if* $|te^t| \leq |\log(c)^{-1}|$.

Proof If $c = 0$, $\{^n c\}_{n \in \mathbb{N}}$ is trivially a 2-cycle, so we can assume without loss of generality that $c \neq 0$. If z is as in Lemma 6.1, then $f^{(2)}(z) = z$, therefore the multiplier of the fixed point z is $f^{(2)}(z) f(z) \log(c)^2 = zc^z \log(c)^2$. The last expression is $te^t \log(c)$ and the lemma follows from fixed point iteration. ∎

Lemma 6.5 excludes Shell and Baker's region N, because there the condition is satisfied, and $\{^n c\}_{n \in \mathbb{N}}$ converges in N, being a 1-cycle. That is, if $c \in N$, then the fixed points of $f(z)$ are given by Lemma 5.1, and $f^{(2)}(z) = f(z) = z$, therefore $|f^{(2)}(z) f(z) \log(c)^2| = |W(-\log(c))^2| = |(\phi^{-1}(c))^2| \leq 1$, by Theorem 6.4.

Calling $M = \{c \in \mathbb{C} \backslash N: |te^t| \leq |\log(c)^{-1}|\}$ the region of Lemma 6.5, M is bounded by the curve $\partial M = \{c \in \mathbb{C} \setminus \partial N: |te^t| = |\log(c)^{-1}|\}$ and is shown in figure 4. ∂N intersects ∂M exactly at M's rightmost point, e^{-e}. The region of Lemma 6.5 is the only region in the Complex plane where $\{^n c\}_{n \in \mathbb{N}}$ can be a 2-cycle, according to Baker and Rippon [27, p. 504]. Color computational maps of M and M's relation with N can be found online at Geisler's *tetration* site [29]. Geisler uses the terminology *pseudocircle* for M.

7. Solving the *p*-th auxiliary complex exponential equation $f^{(p)}(z) = z$

A generalization is now obvious, so we are now ready for the main results of this article.

LEMMA 7.1 *If* $c \in \mathbb{C} \backslash \{0, 1\}$, *then the p-th auxiliary complex exponential equation* $f^{(p)}(z) = z$, *admits the solutions* $\{z, f(z), \ldots, f^{(p-1)}(z)\}$, *where*

$$z = \frac{HW(-\log(c), \ldots, \log(c); \log(c))}{\log(c)} \quad (p \text{ arguments}).$$

Proof Similar to that of Lemma 6.1 and note that if z is a solution then $f^{(p)}(z) = z$, so if $k \in \{1, 2, \ldots, p-1\}$ then $f^{(p)}(f^{(k)}(z)) = f^{(k)}(f^{(p)}(z)) = f^{(k)}(z)$, and the lemma follows. ■

COROLLARY 7.2 *Whenever* $\{{}^n c\}_{n \in \mathbb{N}}$ *is an attracting p-cycle, the limits of the p subsequences* $\{{}^{n+k} c\}_{n \in \mathbb{N}}$, $k \in \{0, 1, \ldots, p-1\}$ *are* $\{z, f(z), \ldots, f^{(p-1)}(z)\}$, *where z is given by Lemma* 7.1.

And finally,

LEMMA 7.3 *If* $c \in \mathbb{C} \setminus N$ *and* $t = HW(-\log(c), \ldots, \log(c); \log(c))$, ($p$ *arguments*), *then* $\{{}^n c\}_{n \in \mathbb{N}}$ *is an attracting p-cycle if* $|te^t \Pi_{k=1}^{p-2} f^{(k)}(e^t)| < |\log(c)^{-p+1}|$ *and can be so only if* $|te^t \Pi_{k=1}^{p-2} f^{(k)}(e^t)| \leq |\log(c)^{-p+1}|$.

Proof Follows from equation (6.1) and fixed point iteration, in a similar way as in Lemma 6.5. ■

Example What is $\{{}^n c\}_{n \in \mathbb{N}}$ if $c = -1 + i$? We can use Lemma 7.3 to check the modulus of the corresponding multipliers t_p for p-cycles. $t_1 = \phi^{-1}(c) = -W(-\log(c)) \doteq 1.13445$, by Theorem 6.4 so it cannot be a 1-cycle (i.e. it cannot converge). $t_2 \doteq 0.80847 + 0.33448i$, and $|t_2 e^{t_2} \log(c)| \doteq 4.67684$, so it cannot be a 2-cycle according to Lemma 6.5. $t_3 \doteq 0.03281 - 0.08534i$ and $|t_3 e^{t_3} f(e^{t_3}) \log(c)^2| \doteq 0.94227$, so by Lemma 7.3 it must be a 3-cycle. Accordingly, by Corollary 7.2, $z = HW(-\log(c), \log(c); \log(c)) / \log(c) \doteq -0.03344 - 0.01884i$, so the 3 separate limits of the 3 subsequences are given as $\{z, f(z), f^{(2)}(z)\}$. This set evaluates to $\{-0.03344 - 0.01884i, 1.02959 - 0.08808i, -1.29112 + 1.19363i\}$. The convergence, which is spiral-like, is shown in figure 5.

COROLLARY 7.4 *If* $c \in N$ *then*

$$\frac{HW(-\log(c), \ldots, \log(c); \log(c))}{\log(c)} = \frac{W(-\log(c))}{-\log(c)}.$$

Proof Inside N, $\{{}^n c\}_{n \in \mathbb{N}}$ is always a 1-cycle, therefore there are no fixed points of period $p > 1$, so all expressions given by Corollary 7.2 must coincide with the fixed points of period 1. ■

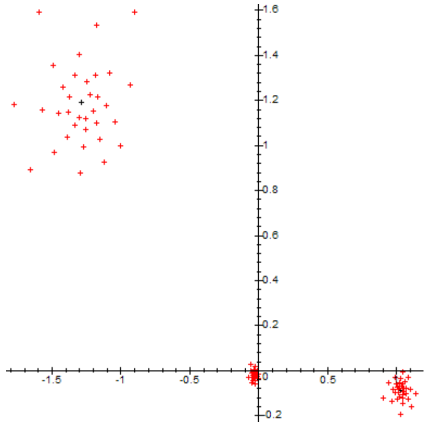

Figure 5. $\{{}^n c\}_{n\in\mathbb{N}}$ for $c = -1 + i$.

We remark that it trivially follows from the above Corollary 7.4 that if $c \in N$, then $HW(-\log(c), \log(c), \ldots; \log(c)) = -W(-\log(c)) = \phi^{-1}(c)$.

8. Equations analytically solvable by HW

Lambert's W function can be used to analytically solve a large class of equations that contain transcendental terms of 'height' or order 1. There are two notable examples, besides the case $c^x = x$, which as we have seen can be solved by W. Two non-trivial examples are the equations $x^n e^x = y$ and $e^x + ax + b = 0$, both of which at first glance seem to be unsolvable through algebraic manipulation.

For the first, the manipulation is relatively easy. $x^n e^x = y \Rightarrow xe^{x/n} = y^{1/n} \Rightarrow (x/n)e^{x/n} = y^{1/n}/n \Rightarrow x/n = W(k, y^{1/n}/n) \Rightarrow x = nW(k, y^{1/n}/n)$, $k \in \mathbb{Z}$.

For the second it is slightly trickier. $e^x + ax + b = 0 \Rightarrow 1 + e^{-x}(ax+b) = 0 \Rightarrow (e^{-b/a}/a)(1+e^{-x}(ax+b)) = 0 \Rightarrow e^{-b/a}/a + e^{-x-b/a}(ax+b)/a = 0 \Rightarrow -e^{-(ax+b)/a}(ax+b)/a = e^{-b/a}/a \Rightarrow -(ax+b)/a = W(k, e^{-b/a}/a) \Rightarrow x = -W(k, e^{-b/a}/a) - b/a$, $k \in \mathbb{Z}$.

The HW functions can be used in a similar manner to solve equations that contain transcendental terms of higher order. For example, the equation $xe^{e^x} = y$, is equivalent to $G(1; x) = y$, which immediately admits the solution $x = HW(1; y)$, while the equation $e^{e^x} = x$ is equivalent to the equation $xe^{-e^x} = 1$, which is equivalent to $G(-1; x) = 1$, and which admits the solution $x = HW(-1; 1) \doteq 0.31813 - 1.33723i$.

One more elaborate example:

$$
\begin{aligned}
2^{7^{3^x}} &= x \Rightarrow \\
x2^{-7^{3^x}} &= 1 \Rightarrow \\
xe^{-\ln(2)e^{\ln(7)e^{x\ln(3)}}} &= 1 \Rightarrow \\
x\ln(3)e^{-\ln(2)e^{\ln(7)e^{x\ln(3)}}} &= \ln(3) \Rightarrow \\
G(-\ln(2), \ln(7); x\ln(3)) &= \ln(3) \Rightarrow \\
G(-\ln(2), \ln(7); w) &= \ln(3).
\end{aligned}
$$

After setting $w = x\ln(3)$. The last equation admits a solution via HW as,

$$w = HW(-\ln(2), \ln(7); \ln(3)) \Rightarrow$$
$$x = \frac{HW(-\ln(2), \ln(7); \ln(3))}{\ln(3)} \doteq 0.35145 + 0.551i.$$

We note here that there is nothing which prevents the c_k from being functions themselves. We can in fact modify Definitions 1.13 and 1.14 as follows:

Let $f_i(z): \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Define $F_{n,m}(z) : \mathbb{N}^2 \times \mathbb{C} \to \mathbb{C}$ as follows.

Definition 8.1

$$F_{n,m}(z) = \begin{cases} e^z & \text{if } n = 1, \\ e^{f_{m-(n-1)}(z)F_{n-1,m}(z)} & \text{if } n > 1. \end{cases}$$

Definition 8.2 $G(f_1(z), f_2(z), \ldots, f_k(z); z) = zF_{k+1,k+1}(z)$.

For simplicity we write $G(f_1, f_2, \ldots, f_k; z)$, and $HW(f_1, f_2, \ldots, f_k; y)$ omitting the notation for the arguments of the functions $f_i(z)$.

LEMMA 8.3 *If $f_i(z): \mathbb{C} \to \mathbb{C}$, $i \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, take values from some domain $\mathfrak{D} \subseteq \mathbb{C}$ and are such that $G(f_1, f_2, \ldots, f_k; z)$ is defined and holomorphic in $\mathfrak{D}$, and if $\mathrm{d}G/\mathrm{d}z \neq 0$ for all $z \in \mathfrak{D}$ then $HW(f_1, f_2, \ldots, f_k; y)$ exists there.*

Proof Follows from Theorem 2.1. Care has to be taken to ensure that $G(\ldots; z)$ is analytic in $\mathfrak{D}$. For example, if $f_1 = z$ and $f_2 = 1/z$, $\mathfrak{D}$ cannot contain any full neighbourhood of either 0, (as $1/z$ is not analytic there), or 0.65904 as this is one of the positive roots of the equation $ze^z - e^z + z = 0$ and that is where $\mathrm{d}G(f_1, f_2; z)/\mathrm{d}z = 0$.

The modified HW functions can now be used to solve transcendental equations with even more complex terms, with just a bit more effort.

Example 1 Consider the equation $2^{3x5^x} = x$, which is unsolvable in terms of elementary functions.

$$\begin{aligned} 2^{3x5^x} &= x \Rightarrow \\ xe^{-3x\ln(2)e^{x\ln(5)}} &= 1 \Rightarrow \\ x\ln(5)e^{-3x\ln(2)e^{x\ln(5)}} &= \ln(5) \Rightarrow \\ x\ln(5)e^{-3x(\ln(5)/\ln(5))\ln(2)e^{x\ln(5)}} &= \ln(5) \Rightarrow \\ we^{-3w(\ln(2)/\ln(5))e^w} &= \ln(5) \end{aligned}$$

after setting $w = x\ln(5)$. The last equation admits the solution

$$w = HW\left(-3x\frac{\ln(2)}{\ln(5)}; \ln(5)\right) \Rightarrow$$
$$x = \frac{HW(-3x(\ln(2)/\ln(5)); \ln(5))}{\ln(5)} \doteq 0.16335 - 0.49553i.$$

Example 2 Let $f_i = x$ for all $i \in \mathbb{N}$ and define the sequence of functions $G_k(x) = G(f_1, f_2, \ldots, f_k; x)$. Note that $G_k(x) = x \cdot {}^k(e^x)$, using Maurer's notation 1.1. For each $k \in \mathbb{N}$, $G_k(0) = 0$, $G_k(x)$ is continuous and strictly increasing throughout $[0, +\infty)$, and $\lim_{x\to+\infty} G_k(x) = +\infty$, therefore $G_k(x)$ is a bijection of $[0, +\infty)$ onto itself, thus for $x \in [0, +\infty)$ each $G_k(x)$ is uniquely invertible and furthermore, the corresponding inverse $HW_k(y) = HW(f_1, f_2, \ldots, f_k; y)$ is real valued in this range. In [8, p. 775] we apply Lemma 5.1 to show that whenever ${}^\infty(e^z)$ exists, it equals $W(-z)/(-z)$. Since $\{{}^n x\}_{n\in\mathbb{N}}$ converges for $x \in [e^{-e}, e^{e^{-1}}]$, it follows that $\{{}^n(e^x)_{n\in\mathbb{N}}\}$ converges for $x \in [0, e^{-1}]$ in particular, so the functions $G_k(x) = x \cdot {}^k(e^x)$ converge (pointwise) to $G_\infty(x) = -W(-x)$, there. The convergence is actually uniform: all functions are of the form $G_k(x) = xe^A$, so they are differentiable everywhere in $[0, e^{-1}]$, with $G_k(0) = 0$ for all $k \in \mathbb{N}$. A simple inductive argument shows that if $m > k$ then $\mathrm{d}G_m(x)/\mathrm{d}x > \mathrm{d}G_k(x)/\mathrm{d}x$ for all $x \in (0, e^{-1}]$ therefore if $m > k$ and if $x_1, x_2 \in (0, e^{-1}]$ with $x_1 < x_2$, then

$$\frac{G_m(x_2) - G_m(x_1)}{x_2 - x_1} > \frac{G_k(x_2) - G_k(x_1)}{x_2 - x_1}$$

from which follows $G_m(x_1) - G_k(x_1) < G_m(x_2) - G_k(x_2)$. Taking limits as $m \to \infty$, we get $G_\infty(x_1) - G_k(x_1) \le G_\infty(x_2) - G_k(x_2)$. In particular, for all $x \in (0, e^{-1}]$, $G_\infty(x) - G_k(x) \le G_\infty(e^{-1}) - G_k(e^{-1}) = -W(-e^{-1}) - G_k(e^{-1}) = 1 - G_k(e^{-1})$, by Lemma 1.8. Consequently, for all $x \in [0, e^{-1}]$, given $\epsilon > 0$ it suffices to pick the n_0 that works at the endpoint $x = e^{-1}$. Then, if $k > n_0$, for all $x \in [0, e^{-1}]$ we have $|G_\infty(x) - G_k(x)| \le |G_\infty(e^{-1}) - G_k(e^{-1})| = |1 - G_k(e^{-1})| < \epsilon$.

Note that $G_\infty(e^{-1}) = 1$, so it follows that the functions $HW_k(y)$ converge uniformly to $HW_\infty(y)$ in $[0, 1]$. But then $HW_\infty(y)$ has to be the inverse of the uniform limit $G_\infty(x) = -W(-x)$ of $G_k(x)$. Solving $-W(-x) = y$ for y, we get $-W(-x) = y \Rightarrow W(-x) = -y \Rightarrow W(-x)e^{W(-x)} = -x = -ye^{-y}$ using W's Definition (1.3), therefore $HW_\infty(y) = ye^{-y}$. Note that this is to be expected, since this is $\log(z)$ whenever $\{{}^n z\}_{n\in\mathbb{N}}$ converges, by Baker and Rippon's Theorem 6.3. Informally thus, whenever $0 \le y \le 1$, $HW_\infty(y) = HW(x, x, \ldots; y) = ye^{-y}$.

9. Derivatives and integrals

Expressions for the derivatives of HW, whenever they exist, follow by applying the expression for the derivative of the inverse of a function (see for example Spivak [22, p. 208]). Simple calculations for example using the Definition (1.6) show that,

$$\frac{\mathrm{d}}{\mathrm{d}y} HW(y) = \frac{HW(y)}{y(1 + HW(y))}$$

which is the same as the derivative of W, as given in Corless *et al.* in [7, p. 2], since $HW(y) = W(y)$. For the general $HW(c_1, c_2, \ldots, c_k; y)$ function, omitting the variable list of constants and remembering (1.6), the expressions are similar:

$$\frac{\mathrm{d}}{\mathrm{d}y} HW(\ldots; y) = \frac{HW(\ldots; y)}{y\Big(1 + HW(\ldots; y)\prod_{k=1}^{k} A_k(y)\Big)} = \frac{HW(\ldots; y)}{y\Big(1 + \prod_{k=1}^{k+1} A_k(y)\Big)}$$

with the $A_k(y)$ defined recursively by $A_{k+1}(y) = \log(A_k(y)/c_k)$, and with $A_1(y) = \log(y/HW(\ldots;y))$.

One can also provide for expressions that relate the integral of HW to that of G, whenever the corresponding functions are real valued. (See for example Parker [30, pp. 439–440] and Spivak [22, p. 235].) Therefore, wherever HW is real valued, and $c_k \in \mathbb{R}$,

$$\int^{y} HW(\ldots;y)\mathrm{d}y = yHW(\ldots;y) - \int^{HW(\ldots;y)} G(\ldots;x)\mathrm{d}x. \tag{9.1}$$

Applying (9.1) with W, we find for example (as in Dubinov [31, p. 32]), that

$$\int W(y)\mathrm{d}y = y\left(W(y) - 1 + \frac{1}{W(y)}\right) + C \tag{9.2}$$

while applying (9.1) with $HW(c;y)$, $c \in \mathbb{R}$ we find,

$$\begin{aligned}\int HW(c;y)\mathrm{d}y &= yHW(c;y) - \int^{HW(c;y)} xe^{ce^x}\mathrm{d}x \\ &= yHW(c;y) - \int^{HW(c;y)} x\sum_{n=0}^{\infty}\frac{c^n e^{nx}}{n!} \\ &= yHW(c;y) - \frac{HW(c;y)^2}{2} - \sum_{n=1}^{\infty}\frac{c^n e^{nHW(c;y)}(nHW(c;y)-1)}{n!n^2}\end{aligned}$$

For the more general functions $HW(f_1, f_2, \ldots, f_k; y)$ the expression for the derivatives is similar. Omitting the variable list of functional parameters from both G and HW, and remembering definition (8.2) we find

$$\frac{\mathrm{d}}{\mathrm{d}z}G(\ldots;z) = F_{k+1,k+1}(z)\,(1 + z\frac{\mathrm{d}}{\mathrm{d}z}\log(F_{k+1,k+1}))$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}y}HW(\ldots;y) = \frac{HW(\ldots;y)}{y\Big[1 + HW(\ldots;y)\,\big[(\mathrm{d}/\mathrm{d}z)\log(F_{k+1,k+1}(z))\big]_{z=HW(\ldots;y)}\Big]}$$

For example, if $f_1 = x$ then

$$\frac{\mathrm{d}}{\mathrm{d}y}HW(x;y) = \frac{HW(x;y)}{y\big[1 + HW(x;y)e^{HW(x;y)} + HW(x;y)^2 e^{HW(x;y)}\big]}$$

We finish with one example which follows from (9.1). Let $f_i = x$ for all $i \in \mathbb{N}$ and define the sequence of functions $G_k(x) = G(f_1, f_2, \ldots, f_k; x)$ and their inverses

$HW_k(y)=HW(f_1,f_2,\ldots,f_k;y)$ in $[0,+\infty)$ again as in Example 2 of the previous section. In [8, p. 776] we show the following corollary:

COROLLARY 9.1 *For* $m\in\mathbb{N}$, ${}^m(e^z)$ *is entire, with series expansion*

$${}^m(e^z)=\sum_{n=0}^{m}\frac{(n+1)^n}{(n+1)!}z^n+\sum_{n=m+1}^{\infty}a_{m,n}z^n$$

where $a_{m,n}$ *are given by the recursion*:

$$a_{m,n}=\begin{cases}1 & \text{if } n=0,\\ \dfrac{1}{n!} & \text{if } m=1,\\ \dfrac{\sum_{j=1}^{n} j a_{m,n-j}a_{m-1,j-1}}{n} & \text{otherwise.}\end{cases}\tag{9.3}$$

Therefore, using equation (9.1) and Corollary 9.1,

$$\begin{aligned}\int HW_k(y)\mathrm{d}y &= yHW_k(y)-\int^{HW_k(y)}G_k(x)\mathrm{d}x\\ &= yHW_k(y)-\int^{HW_k(y)}x\cdot{}^k(e^x)\mathrm{d}x\\ &= yHW_k(y)-\int^{HW_k(y)}\left(\sum_{n=0}^{k}\frac{(n+1)^n}{(n+1)!}x^{n+1}+\sum_{n=k+1}^{\infty}a_{k,n}x^{n+1}\right)\mathrm{d}x\\ &= yHW_k(y)-\sum_{n=0}^{k}\frac{(n+1)^n}{(n+1)!(n+2)}HW_k(y)^{n+2}-\sum_{n=k+1}^{\infty}\frac{a_{k,n}}{n+2}HW_k(y)^{n+2}.\end{aligned}$$

with $a_{k,n}$ given by recursion (9.3). As we pointed out in Example 2 the convergence of $G_k(x)$ and $HW_k(y)$ is uniform whenever $x\in[0,e^{-1}]$ and $y\in[0,1]$, so remembering that the uniform limits are $G_\infty(x)=-W(-x)$ and $HW_\infty(y)=ye^{-y}$ there, using equation (9.3), we get,

$$\begin{aligned}\int HW_\infty(y)\mathrm{d}y &= yHW_\infty(y)-\int^{HW_\infty(y)}G_\infty(x)\mathrm{d}x\Rightarrow\\ \int ye^{-y}\mathrm{d}y &= y^2e^{-y}-\int^{ye^{-y}}-W(-x)\mathrm{d}x\\ &= y^2e^{-y}-\left[-x(W(-x)-1+W(-x)^{-1})\right]_{x=ye^{-y}}\\ &= y^2e^{-y}-\left[-ye^{-y}(W(-ye^{-y})-1+W(-ye^{-y})^{-1})\right]\\ &= y^2e^{-y}-\left[-ye^{-y}(-y-1+(-y)^{-1})\right]\\ &= y^2e^{-y}-y^2e^{-y}-ye^{-y}-e^{-y}\\ &= -ye^{-y}-e^{-y}\end{aligned}$$

which is to be expected, since using integration by parts,

$$\int ye^{-y}\mathrm{d}y=-ye^{-y}-e^{-y}.$$

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Appendix A: a Maple algorithm for the principal branch of $HW(\ldots;y)$

```
digits:=40;
HW:=proc()
#call as HW(c1,c2,...ck,y,n) or as HW(f1,f2,...fk,y,n)
#for example HW(3, -log(2),12,30) or HW(z,z^2,12,30)
#y: actual argument. n: for approximate zero
local y,n,c,s,p,sol,i,aprx,dy,dist,r,oldr,newr,fun,dfun, eps;
if nargs<2 then ERROR('At least two arguments required.') fi;
n:=args[-1]; #recover last argument
y:=args[-2]; #recover second to last argument
c:=[args[1..-3]]; #recover constants or functions of G
if y=0 then 0 #G(...;0)=0
else
  dist:=infinity;
  eps:=1e-32;
  fun:= exp(z); #build G(...;z) recursively
  for i from 1 to nargs-2 do fun:= exp(c[-i]*fun) od;
  fun:= z*fun-y;
  dfun:=diff(fun,z); #dG/dz
  s:=series(fun,z,n); #convert to series
  p:=convert(s,polynom); #convert to polynom
  sol:={fsolve(p=0,z,complex)}; #find all roots
  for i from 1 to nops(sol) do #pick best root
    aprx:=evalf(subs(z=op(i,sol),fun));
    dy:=evalf(abs(aprx));
    if dy<=dist then
      r:=op(i, sol);
      dist:=dy;
    fi;
  od;
#start Newton with approximate zero found.
  oldr:=r;
  newr:=r-evalf(subs(z=r, fun)/subs(z=r, dfun));
  for i from 1 to 1000 while (abs((oldr-newr)/oldr)>eps) do
    oldr:=newr;
    newr:=newr-evalf(subs(z=newr,fun)/subs(z=newr,dfun));
    od;
    newr;
  fi;
end:
```

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