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TRANSFINITE ORDINALS IN RECURSIVE NUMBER THEORY

R. L. GOODSTEIN

The possibility of constructing a numerical equivalent of a system of transfinite ordinals, in recursive number theory, was briefly indicated in a previous paper,1 where consideration was confined to ordinals less than $\varepsilon$ (the first to satisfy $\varepsilon = \omega^\omega$). In the present paper we construct a representation, by functions of number-theoretic variables, for ordinals of any type.

In addition to definite numerals, and numeral variables, we introduce majorant variables $\sigma, \omega, \omega_r, r \geq 1$. A relation containing a single majorant variable $\sigma$ is required to hold, not necessarily for all non-negative integral values of $\sigma$, but for all values greater than some assigned constant. A relation $R(\omega, \omega_1, \omega_2, \cdots, \omega_m)$ between the majorant variables $\omega, \omega_r, (1 \leq r \leq m)$, holds if there is a constant $c_0$ and recursive functions $c_r(n_0, n_1, \cdots, n_{r-1})$ such that $R(n_0, n_1, n_2, \cdots, n_m)$ holds for all non-negative integers $n_0, n_1, \cdots, n_m$ such that $n_0 \geq c_0$ and $n_{r+1} \geq c_{r+1}(n_0, n_1, \cdots, n_r), 0 \leq r \leq m - 1$. For instance, if $a, b, c$, are definite numerals then $\sigma^a > a \sigma^b + c$, since $n^a > an^b + c$ if $n > \max(a, b, c)$; in particular, for any definite numeral $a, \sigma > a$. An example of a relation between two majorant variables is $\omega^{\omega_1} + 3\omega^2 + 4 > \omega^3_1 + 7\omega^2 + 11$ which holds since $n^a N^b + 3n^2 + 4 > n^3_1 N^2 + 7n^2 + 11$ when $n \geq 12$ and $N \geq n^3$.

Majorant variables have the same algebra as numeral variables, that is to say, they obey the same rules of addition, multiplication, and exponentiation, for if a relation $R(n_0, n_1, n_2, \cdots, n_m)$ holds for all non-negative integers $n_r, 0 \leq r \leq m$, without restriction, then a fortiori, $R(\omega, \omega_1, \cdots, \omega_m)$ holds.

We commence with a generalization of the concept of a numeral expressed with assigned digits in a scale of notation. If $f(x)$ is any function of a numeral variable $x$ (and possibly of other numeral or majorant variables as well), such that $f(0) = 0, f(1) = 1$ and $f(x + 1) \geq f(x) + 1$ for all $x$, we define, for any numerals $k \geq 1, n \geq 0$ and (a numeral or majorant variable) $\sigma > f(k)$, the function $\phi_{k, n}(x)(\sigma)$ recursively as follows:

If $(k + 1)^a$ is the greatest power of $k + 1$ not exceeding $n$, and $c(k + 1)^a$ is the greatest multiple of $(k + 1)^a$ not exceeding $n$, then

\begin{align*}
\phi_{k, n}^{\langle 0 \rangle}(0) &= 0 \quad \text{(i)} \\
\phi_{k, n}^{\langle \sigma \rangle}(n) &= f(c)\phi_{k, n}^{\langle \sigma \rangle}(n) + \phi_{k, n}^{\langle \sigma \rangle}(n - c(k + 1)^a), n \geq 1. \quad \text{(ii)}
\end{align*}

For given $k$ and $n$ the equations (i), (ii) determine the 'value' of $\phi_{k, n}(x)(\sigma)$ as a function of $x$ and $f(0), f(1), \cdots, f(k)$ which we call the representation of $n$ with digits $f(r), 0 \leq r \leq k$, and base $\sigma$. For instance $\phi_{k, n}^{\langle 2 \rangle}(561) = f(2)\sigma^2 + f(3)\sigma^{(2)} + 1$.

We shall write either $x$ or $y$ for the variable in the affix function $f(x)$, and when $f$ contains more than one argument the operative variable will be written in the last argument place; thus the operative variable in $h(p, q, x)$ is $x$, in the last argument place. We note that since $f(0) = 0$ and $f(x + 1) \geq f(x) + 1$ therefore

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$f(x) \geq x$ for any numeral $x$. The 'value' of $\phi_{m+1}^r(n)$ for a given $n$ is of course the familiar representation of $n$ in the scale $m + 1$, with digits $0, 1, 2, \cdots, m$.

**Theorem 1.** $\phi_{k \sigma}^{(r)}(r) = f(r)$, $(r \leq k)$.

For the exponent of the greatest power of $k + 1$, not exceeding $r$, is zero, and the greatest multiple of $(k + 1)^{r}$, not exceeding $r$, is $r$ itself. Hence

$$\phi_{k \sigma}^{(r)}(r) = f(r)\sigma^{0} + \phi_{k \sigma}^{(r)}(0) = f(r).$$

Hence

$$\phi_{k \sigma}^{(r)}(k + 1) = f(1)\sigma^{\phi_{k \sigma}^{(r)}(1)} + \phi_{k \sigma}^{(r)}(0) = \sigma, \quad (k \geq 1).$$

**Theorem 2.** For a fixed numeral $k \geq 1$, and a definite numeral (or majorant variable) $s$ satisfying $s \geq f(k + 1), \phi_{k \sigma}^{(x)}(n)$ is strictly monotonic increasing with $n$.

For brevity we write $\psi(n)$ for $\phi_{k \sigma}^{(x)}(n)$. We have to prove, for all $n \geq 0$, $\psi(n + 1) \geq \psi(n) + 1$.

By theorem 1, since $f(n + 1) \geq f(n) + 1$, the inequality holds for $n \leq k$; suppose that it holds for $n \leq m - 1$, where $m \geq k + 1$. Let $c$ be the greatest integer such that $(k + 1)^c \leq m$ and $b$ the greatest integer such that $b(k + 1)^c \leq m$, and let $a = m - b(k + 1)^c$, so that $1 \leq c$, $1 \leq b < k + 1$ and $0 \leq a < (k + 1)^c$.

Then $\psi(m) = f(b)s^{\psi(c)} + \psi(a)$.

If $a = 0, m + 1 = b(k + 1)^c + 1$ and so $\psi(m + 1) = f(b)s^{\psi(c)} + 1 = \psi(m) + 1$, since $\psi(0) = 0$.

If $a > 0$ and $a + 1 < (k + 1)^c$, then

$$\psi(m + 1) = f(b)s^{\psi(c)} + \psi(a + 1) \geq f(b)s^{\psi(c)} + \psi(a) + 1 = \psi(m) + 1,$$

for $a + 1 < (k + 1)^c \leq m - 1$, and so, by hypothesis, $\psi(a + 1) \geq \psi(a) + 1$.

If $a + 1 = (k + 1)^c$, then $m + 1 = (b + 1)(k + 1)^c$.

Consider first the case $b + 1 < k + 1$. Since $a < a + 1 = (k + 1)^c < m$, therefore $\psi(a + 1) \leq \psi(a + 1) = \psi(k + 1)^c = s^{\psi(c)}$, and so

$$\psi(m + 1) = f(b + 1)s^{\psi(c)} \geq f(b)s^{\psi(c)} + s^{\psi(c)} \geq f(b)s^{\psi(c)} + \psi(a) + 1 = \psi(m) + 1.$$

If $b + 1 = k + 1$, then $m + 1 = (k + 1)^{c + 1}$. Since $m > (k + 1)^c \geq 1 + ck \geq 1 + c$, therefore, by hypothesis, $\psi(c + 1) \geq \psi(c) + 1$.

Moreover $s \geq f(b) + 1$ and $a + 1 = (k + 1)^c < m$, so that $s - f(b)s^{\psi(c)} \geq s^{\psi(c)} = \psi(a + 1) \geq \psi(a) + 1$; hence

$$\psi(m + 1) = s^{\psi(c + 1)} \geq s^{\psi(c)} \cdot s \geq f(b)s^{\psi(c)} + \psi(a) + 1 = \psi(m) + 1,$$

which completes the proof of the inequality for $n = m$, and so, by induction, theorem 2 holds for all $n$.

We notice two special cases of theorem 2:

2.1 Taking $f(x) = x$, we have, if $\sigma > k \geq 1$, $\phi_{k \sigma}^x(n)$ is strictly increasing with $n$.

2.2 Taking $f(x) = \phi_{p, q + 1}^x(x), q \geq p \geq 1$, we have, if $\sigma > \phi_{p, q + 1}^x(k), \phi_{p, q + 1}^x(n)$ is strictly increasing with $n$. 
Theorem 3. If \( f(x), g(x) \) are strictly monotonic increasing, and \( p \geq g(q) \), and \( s > f(p) \) (where \( s \) is a numeral or a majorant variable), then, for all \( n \), and \( q \geq 1 \)

\[
\phi^{f(s)}_{p,s} \{ \phi^{g(q)}_{q,p+1}(n) \} = \phi^{f(g(q))}_{q,s}(n).
\]

For \( n = 0 \), each expression takes the value zero; if the equality holds for \( n = 0, 1, 2, \ldots, m - 1 \) then we deduce that it holds for \( n = m \geq 1 \). Let \((q + 1)^a\) be the greatest power of \( q + 1 \) and \( c(q + 1)^a \) the greatest multiple of \((q + 1)^a\) not exceeding \( m \), and let \( b = m - c(q + 1)^a \), so that \( c < q + 1, b < (q + 1)^a \), and \( m < (q + 1)^{a+1} \). Then

\[
\phi^{g(q)}_{q,p+1}(m) = g(c)(p + 1)^{\phi^{f(s)}_{q,p+1}(a)} + \phi^{g(q)}_{q,p+1}(b)
\]

\[
< g(c)(p + 1)^{\phi^{f(s)}_{q,p+1}(a)} + (p + 1)^{\phi^{f(s)}_{q,p+1}(a)}
\]

by theorem 2, since \( b < (q + 1)^a \),

\[
= \{g(c) + 1\}(p + 1)^{\phi^{f(s)}_{q,p+1}(a)} \leq (p + 1)^{\phi^{f(s)}_{q,p+1}(a)+1}
\]

since \( g(c) \leq g(q) < p + 1 \).

Thus \( \phi^{g(q)}_{q,p+1}(a) \) is the exponent of the greatest power of \( p + 1 \) and \( g(c)(p + 1)^{\phi^{g(q)}_{q,p+1}(a)} \) is the greatest multiple of \((p + 1)^{\phi^{g(q)}_{q,p+1}(a)}\) not exceeding \( \phi^{g(q)}_{q,p+1}(m) \).

Therefore

\[
\phi^{f(s)}_{p,s} \{ \phi^{g(q)}_{q,p+1}(m) \} = f(g(c))\phi^{\phi^{f(s)}_{q,s}}_{q,s} \{ \phi^{g(q)}_{q,p+1}(a) \} + \phi^{f(s)}_{p,s} \{ \phi^{g(q)}_{q,p+1}(b) \}
\]

\[
= f(g(c))\phi^{\phi^{f(s)}_{q,s}}_{q,s} (a) + \phi^{f(g(q))}_{q,s}(b)
\]

(by hypothesis, since \( a < m, b < m \))

\[
= \phi^{f(g(q))}_{q,s}(m),
\]

which completes the inductive proof.

In terms of two functions \( p(n), p'(n) \) we define, for \( k \geq 0, n \geq 0 \), (anticipating theorem 4 below)

\[
X_{p,p'}(0, k) = k
\]

\[
X_{p,p'}(n + 1, k) = \phi^{X_{p,p'}(n+1)}_{p,n}(n+1)(k)
\]

provided \( p(n) \geq 1 \) and \( p'(n) \geq X_{p,p'}(n, p(n)) \), for all \( n \).

Theorem 4. For all \( n \geq 0 \), \( X_{p,p'}(n, k) \) is strictly monotonic increasing with \( k \).

The theorem holds for \( n = 0 \), by definition of \( X_{p,p'}(0, k) \), and if it holds for \( n = m \), then it holds for \( n = m + 1 \) by theorem 2, whence by induction, the theorem is true for all \( n \).

Since \( X_{p,p'}(n, 0) = 0 \), it follows that, for all \( k \geq 0 \), \( X_{p,p'}(n, k) \geq k \).
THEOREM 5. If \( p(r), q(r) \geq 1 \), and if \( p'(r) \geq \max \{ X_{p,p'}(r, p(r)), X_{q,p}(r, q(r)) \} \)
and \( q(r) \geq X_{p,q}(r, p(r)), 0 \leq r \leq n - 1 \), then \( X_{p,p'}(n, k) = X_{q,p}(n, X_{p,q}(n, k)), n \geq 0, k \geq 0 \).

The theorem holds for \( n = 0 \), since \( X_{\lambda,\mu}(0, k) = k \) for any \( \lambda, \mu \); if it holds for \( n = m \), then

\[
X_{q,p'}(m + 1, X_{p,q}(m + 1, k)) = \phi_{q,p'}(m + 1, X_{p,q}(m + 1, k)) \\
= \phi_{q,p'}(m + 1, (\phi_{p,q}(m + 1, k)) \\
= \phi_{p,q}(m + 1, k) \]

by theorem 3,

which proves the theorem for \( n = m + 1 \), and so for all values of \( n \).

THEOREM 6. If \( p(k) \geq 1, q(k) \geq 1 \), and if \( r(k) \geq \max \{ X_{p,r}(k, p(k)), X_{q,r}(k, q(k)) \} \) and \( s(k) \geq \max \{ X_{p,s}(k, p(k)), X_{q,s}(k, q(k)) \} \), \( n > k \geq 0 \), then

\[
X_{p,r}(n, p(n)) \leq X_{q,r}(n, q(n))
\]

according as \( X_{p,s}(n, p(n)) \leq X_{q,s}(n, q(n)) \).

For \( 0 \leq k \leq n - 1 \), let

\[
t(k) = \max \{ X_{p,r}(k, p(k)), X_{q,r}(k, q(k)), X_{r,s}(k, r(k)), X_{s,t}(k, s(k)) \}
\]

then by theorem 5,

\[
X_{r,s}(n, X_{p,r}(n, p(n)) = X_{r,s}(n, p(n))
\]

and

\[
X_{r,s}(n, X_{q,r}(n, q(n))) = X_{r,s}(n, q(n)),
\]

and so, by theorem 4,

\[
X_{p,s}(n, p(n)) \leq X_{q,s}(n, q(n))
\]

according as

\[
X_{p,r}(n, p(n)) \leq X_{q,r}(n, q(n)).
\]

Similarly

\[
X_{p,s}(n, p(n)) \leq X_{q,s}(n, q(n))
\]

according as

\[
X_{p,r}(n, p(n)) \leq X_{q,r}(n, q(n)),
\]

whence theorem 6 follows.

THEOREM 7. If \( p(n) \geq k > 0 \) and \( p(r) \geq 1, p'(r) \geq X_{p,p'}(r, p(r)) \) for \( 0 \leq r \leq n \), then \( X_{p,p'}(n, k) = X_{p,p'}(n + 1, k), n \geq 0 \).
For \( X_{p,p'}(n + 1, k) = \phi_{x^{p,p'}(n+1)}(k) = X_{p,p'}(n, k) \), by theorem 1. It follows that if \( N > n \) and \( p(m) \geq k \) for \( n \leq m < N \), then \( X_{p,p'}(n, k) = X_{p,p'}(N, k) \).

If \( \omega, \omega_r \ (r > 0) \) are majorant variables, we define, for all \( n \geq 0 \), \( p(n) \geq 1 \), \( k \geq 0 \) (anticipating theorem 9 below)

\[
\Omega_p(0, k) = k
\]

\[
\Omega_p(n + 1, k) = \phi_{x^{p(n)}(\omega_n)}(k), \quad \omega_0 = \omega.
\]

(ii)

For given \( k, n \) and \( p(r), 0 \leq r \leq n - 1 \), the equations (i) and (ii) determine the 'value' of \( \Omega_p(n, k) \) as a function of \( \omega, \omega_r, 1 \leq r \leq n - 1 \), which we call a transfinite ordinal of type \( n \).

**Theorem 8.** A transfinite ordinal of type \( n \) is also of type \( n + 1 \).

For, by theorem 1, if \( p(n) \geq k \),

\[
\Omega_p(n + 1, k) = \phi_{x^{p(n)}(\omega_n)}(k) = \Omega_p(n, k).
\]

It follows that, if \( p(m) \geq k \) for \( n \leq m \leq N - 1 \), then

\[
\Omega_p(n, k) = \Omega_p(N, k).
\]

**Theorem 9.** For all \( n \geq 0 \), \( \Omega_p(n, k) \) is monotonic strictly increasing with \( k \).

For \( \Omega_p(0, k) = k \), and if the theorem holds for \( n = m \), then by theorem 2 it holds for \( n = m + 1 \).

It follows that if \( k > p(n) \geq \lambda, \Omega_p(n + 1, k) > \Omega_p(n + 1, \lambda) = \Omega_p(n, \lambda) \), by theorem 8, and so there are ordinals of type \( n + 1 \) which are not of lower type.

**Theorem 10.** If \( p(r) \geq 1 \) and \( q(r) \geq X_{p,q}(r, p(r)), 0 \leq r \leq n - 1 \), then for all \( k \geq 0, n \geq 0, \Omega_p(n, k) = \Omega_q(n, X_{p,q}(n, k)) \).

The theorem holds for \( n = 0 \) since \( \Omega_q(0, X_{p,q}(0, k)) = \Omega_p(0, k) \). If it holds for \( n = m \) then

\[
\Omega_q(m + 1, X_{p,q}(m + 1, k)) = \phi_{x^{q(m)}(\omega_m)}(\phi_{x^{p(q)(m)}(\omega_m)+1}(k))
\]

\[
= \phi_{x^{p(q)(m)}(\omega_m)}(k)
\]

by hypothesis (and using theorems 3, 4 and 9)

\[
= \Omega_p(m + 1, k)
\]

which proves theorem 10 for \( n = m + 1 \), and so for all values of \( n \).

**Theorem 11.** If \( p(r) \geq 1, q(r) \geq 1, 0 \leq r \leq n \), then, for all \( n \geq 0 \)

\[
\Omega_p(n, p(n)) \leq \Omega_q(n, q(n))
\]

according as

\[
X_{p,r}(n, p(n)) \leq \Omega_q(n, X_{p,q}(n, p(n)))
\]

where \( r(k) \geq \max \{X_{p,r}(k, p(k)), X_{q,r}(k, q(k))\}, \quad n > k \geq 0 \).

By theorem 10, \( \Omega_p(n, p(n)) = \Omega_q(n, X_{p,q}(n, p(n))) \) and \( \Omega_q(n, q(n)) = \Omega_q(n, X_{q,q}(n, q(n))) \), whence theorem 11 follows from theorem 9. By theorem 6,
the relation (i) is independent of the choice of the numbers \( r(k) \), provided condition (ii) is satisfied.

Theorem 11 shows that the decreasing ordinal theorem, which asserts that a sequence of decreasing ordinals necessarily terminates, is, for ordinals of any type, equivalent to a number-theoretic proposition.

For the sequence of ordinals \( \Omega_{p_n(i), p_n(i)} \), \( n = 0, 1, 2, \ldots \) is steadily decreasing if

\[
X_{p_{n+1}, r_n(i), p_{n+1}(i)} < X_{p_n, r_n(i), p_n(i)}
\]

where

\[
\begin{align*}
    r_n(k) &\geq \max \{ X_{p_n, r_n(k), p_n(k)}, X_{p_{n+1}, r_n(k), p_{n+1}(k)} \}, \quad n \geq 0, \quad 0 \leq k < i.
\end{align*}
\]

Thus the decreasing ordinal theorem is equivalent to the proposition:

If for all \( n \) and \( 0 \leq k < i \), \( p_n(k) \geq 1 \) and \( p_n(i) \geq 0 \), and if

\[
- r_n(k) = \max \{ X_{p_n, r_n(k), p_n(k)}, X_{p_{n+1}, r_n(k), p_{n+1}(k)} \}
\]

and

\[
X_{p_{n+1}, r_n(i), p_{n+1}(i)} < X_{p_n, r_n(i), p_n(i)} \quad \text{when} \quad p_n(i) > 0,
\]

then there is a value of \( n \) for which \( p_n(i) = 0 \).

**Example.** If \( p(0) = 1, p(1) = 4 \) and \( q(0) = 2, q(1) = 99 \) then \( \Omega_{p}(2, 116) = \Omega_q(2, 27.100^3 + 4.100 + 1) = \omega^\omega \omega_1^\omega + (\omega + 1) \omega_1 + 1 \).

For \( 116 = 4.5^3 + 3.5 + 1 \) and so

\[
\begin{align*}
    \Omega_p(2, 116) &\equiv \phi^5_{3,\omega}(4) \omega_1^{\omega_1^{\omega_1^{\omega_1}}} + \phi^5_{3,\omega}(3) \omega_1 + \phi^5_{3,\omega}(1) \\
    &\equiv \omega^\omega \omega_1^\omega + (\omega + 1) \omega_1 + 1,
\end{align*}
\]

and

\[
\begin{align*}
    \Omega_q(2, 27.100^3 + 4.100 + 1) &\equiv \phi^5_{3,\omega}(27) \omega_1^{\omega_1^{\omega_1^{\omega_1}}} + \phi^5_{3,\omega}(4) \omega_1^{\omega_1^{\omega_1}} + \phi^5_{3,\omega}(1),
\end{align*}
\]

There are recursive functions, with transfinite ordinal 'values,' of greater generality than \( \Omega_p(n, k) \). The function \( \Omega \) is based on the three processes of addition, multiplication, and exponentiation, but obviously functions based on a wider range of processes may be defined recursively. If we confine our attention just to two processes, addition and multiplication, then ordinals of type one will be polynomials in \( \omega \), each term having the form \( \omega \omega \cdot \cdots \omega \), and so only ordinals less than \( \omega^\omega \) will be determined; introducing exponentiation greatly extends the class of ordinal functions, and correspondingly the introduction of additional processes effects further extensions.

We define, for \( k \geq 0, a \geq 0, n \geq 0 \)

\[
G(0, a, n) = n + 1, \quad G(1, a, 0) = a, \quad G(2, a, 0) = 0, \quad G(k + 3, a, 0) = 1, \quad G(k + 1, a, n + 1) = G(k, a, G(k + 1, a, n)).
\]

Then \( G(1, a, n) = a + n, G(2, a, n) = na, G(3, a, n) = a^n \) (proof by induction), so that for \( k = 1, 2, 3 \) the function \( G(k, a, n) \) defines addition, multiplication, and
exponentiation. For \( k \geq 4 \), \( G(k, a, n) \) defines successive new processes (which we may call tetration, pentation, hexation, and so on). For instance

\[ G(4, a, n) = a^a, \]

the expression containing \( n \) \( a \)'s. It is convenient to extend the use of the term index so that \( n \) is called the index of tetration in \( G(4, a, n) \), the index of pentation in \( G(5, a, n) \), and so on. Simple notations for \( G(4, a, n) \), \( G(5, a, n) \), \( G(6, a, n) \) are (by analogy with exponentiation) \( ^n a, _n a, a_n \), respectively, though this admits of no further extension.

If \( a, b, c \) are chosen in turn to be the greatest integers such that \( ^a (k + 1) \leq n, \ ( ^b (k + 1) )^c \leq n \), \( c ( ^{a} (k + 1) )^b \leq n \), then, for \( k \geq 1 \), we define

(i) \( \phi_{k, a}^{(e)}(n) = f(n), \quad 0 \leq n \leq k \),

(ii) \( \phi_{k, a}^{(e)}(n) = \phi_{k, a}^{(e)}(c)[ \phi_{k, a}^{(e)}(a)]^{ \phi_{k, a}^{(e)}(b)} + \phi_{k, a}^{(e)}(n - c( ^{a} (k + 1))^{b}), \quad n \geq k + 1 \).

We observe that \( c < k + 1 \) and, for \( a \geq 1 \), since \( ( ^a x ) = a^a x \) if \( x^{(a-1)x} = x^{(a^2)} \), that is, if \( n = ^a x / ^{a-1} x \), therefore \( b < ^a (k + 1) / ^{a-1} (k + 1) \).

For given \( k, n \), the 'value' of \( \phi_{k, a}^{(e)}(n) \), determined by the repeated application of equations (i) and (ii), is a function of \( \phi \) and \( f(\gamma) \), \( 0 \leq \gamma \leq k \) which we call the representation of \( n \) with base \( \phi \) and digit \( f(0) = 0, f(1) = 1, f(2), \cdots, f(k) \), using the four operations, addition, multiplication, exponentiation, and tetration. For example

\[ \phi_{1, \omega}^{(e)}(100) = (\omega + 1) \{ ^{\omega+1} \omega \} + (\omega + 1) \{ ^{\omega} \omega \} \{ ^{\omega+1} \omega \} + \omega \{ ^{\omega+1} \omega \} + \omega \{ ^{\omega} \omega \} \].

To determine the representation of \( n \) using \( \lambda \) operations, we define \( \phi_{k, a}^{(e)}(n) \) recursively as follows: For a given sequence \( x(n) \) and for \( k \geq 1 \), we have

\[ g_{k, a}^{(e)}(1) = G(\lambda, k + 1, x(1)), \]

\[ g_{k, a}^{(e)}(m + 1) = G(\lambda - m, g_{k, a}^{(e)}(m), x(m + 1)), \quad 1 \leq m \leq \lambda - 1, \]

and for \( n \geq k + 1 \) we choose \( a(\gamma) \) so that for \( 1 \leq \gamma \leq \lambda, a(\gamma) \) is the greatest integer such that \( g_{k, a}^{(e)}(s) \leq n, 1 \leq s \leq \gamma, (so \ that \ a(\gamma) < n \ for \ 1 \leq \gamma \leq \lambda) \).

Then we define

\[ Y_{k, a}^{(e)}(1) = G(\lambda, \sigma, \phi_{k, a}^{(e)}(a(1))), \]

\[ Y_{k, a}^{(e)}(m + 1) = G(\lambda - m, Y_{k, a}^{(e)}(m), \phi_{k, a}^{(e)}(a(m + 1))), \quad 1 \leq m \leq \lambda - 1, \]

and

\[ \phi_{k, a}^{(e)}(n) = f(n), \quad n \leq k, \]

\[ \phi_{k, a}^{(e)}(n) = Y_{k, a}^{(e)}(\lambda), \quad n \geq k + 1. \]

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