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Ultra power and ultra exponential functions

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Supposing that a is a positive real number, then for each natural number n, the notation $a^{\underline{n}}$ is called *a to the ultra power of* n, and we define by

$$a^{\underline{n}} = a^{a^{\underline{n-1}}}, \quad a^{\underline{1}} = a.$$

In the other words, $a^{\underline{n}} = a^{a^{\cdot^{\cdot^{a}}}}$, n times.

In [Euler, L., De formulis exponentialibus replicatus, *Acta Academiac Petropolitenae*, **1**, 38–60.] the necessary and sufficient condition for the convergence of $\lim_{n\to\infty} a^{\underline{n}}$ is proved by Euler and in [Baker, I.N. and Rippon, P.J., 1983, Convergence of infinite exponentials. Annales Academiac Scentiarium Fennicae Mathematika Series AI., **8**, 179–186] it is studied for the case $a \in \mathbb{C}$. Also in Macdonnell, J. [1989, Some critical points on the hyperpower function ${}^{n}x = x^{x^{\cdot^{\cdot^{\cdot}}}}$ *International Journal of Mathematical Education in Science and Technology*, **20**(2), 297–305.], by supposing $a = x$ be variable, some property of the function $f(x) = \lim_{n\to\infty} x^{\underline{n}}$ is introduced. But, in this paper, we want to introduce and discuss a different topic, considering the following explanation: we can consider the serial operations as production, power and ultra power respectively.

As we know, due to algebraic properties of power, it can be extended from natural to rational numbers (and then other numbers) but this cannot be done for ultra power, because it does not have any useful algebraic properties except $a^{\underline{n}} = a^{a^{\underline{n-1}}}$. Perhaps this is the reason that ultra power is not extended so far. For removing this problem, we first introduce the following functional equation and then study some of its properties

$$f(x) = a^{f(x-1)}, \quad f(0) = 1.$$

(If f be a function satisfying the equation, then $f(n) = a^{\underline{n}}$ for all natural n).

We state and prove a uniqueness theorem about the above functional equation and extend the definition of ultra power by it. Also, we introduce the new functions uxp_a (*ultra exponential functions*) and study their properties.

Keywords: Ultra power; Ultra exponential functions; Natural ultra exponential function; Ultra exponential functional equation; Infra logarithm functions; Convex functions; Uniqueness theorem

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1. Preliminaries

Let $a \neq 1$ be a positive number and n be a natural number. We call the notation $a^{\underline{n}}$ *a to the ultra power of n* and define by the following recursive definition

$$(*_1) \quad a^{\underline{n}} = a^{a^{\underline{n-1}}} \quad \text{for } n = 2, 3, \ldots, \quad a^{\underline{1}} = a.$$

Therefore, $a^{\underline{2}} = a^a$, $a^{\underline{3}} = a^{a^a}$ and so on. Let us apply the relation $(*_1)$ for $n = 1$:

$$a = a^{\underline{1}} = a^{a^{\underline{0}}},$$

so we define $a^{\underline{0}} = 1$. Putting $n = 0$ implies that $a^{\underline{-1}} = 0$. But, it is impossible to take $n = -1$ and so $a^{\underline{-2}}, a^{\underline{-3}}, \ldots$ are not well defined.

DEFINITION 1.1 *Let $0 < a \neq 1$ and n be an integer such that $n \geq -1$. We define $a^{\underline{n}}$ by $a^{\underline{-1}} = 0$, $a^{\underline{0}} = 1$ and*

$$a^{\underline{n}} = a^{a^{\cdot^{\cdot^{a}}}}, \quad n \text{ times}.$$

In order to extend the definition of the ultra power to other real values, we cannot apply the known method about the power, because ultra power does not have any useful algebraic property [except that $(*_1)$]. For this reason, in view of $(*_1)$ consider the following functional equation

$$(*_2) \quad f(x) = a^{f(x-1)} \text{ for all } x > -1.$$

If f satisfies $(*_2)$ and $f(0) = 1$, then $f(n) = a^{\underline{n}}$ for $n = -1, 0, 1, \ldots$ Hence, we call the above functional equation with $f(0) = 1$ the *ultra exponential functional equation*. We show that $(*_2)$ [with $f(0) = 1$] has infinitely many solutions below (Corollary 2.3), therefore, uniqueness conditions for the ultra exponential functional equation are needed. Hence, we introduce some notations, then in section 2 we will state a uniqueness theorem for ultra power and prove it.

Let f be a real value function and $n \in \mathbb{Z}$. Put

$$f^n = \begin{cases} fo\cdots of \ (n \text{ times}); & n \in \mathbb{N}^* \\ \mathrm{I}; & n = 0 \\ f^{-1}o\cdots of^{-1} \ (-n \text{ times}); & n \in \mathbb{Z}^-, \end{cases}$$

when compositions are possible.

If $0 < a \neq 1$, then $\exp_a(x) = a^x$ and $\exp(x) = \exp_e(x) = e^x$ (for all x) and $\exp_a^{-1} = \log_a$, $\ln = \exp^{-1}$. Denote by $[x]$ the largest integer not exceeding x and $(x) = x - [x]$.

2. A uniqueness theorem for ultra exponential functional equation and extending the definition of ultra power

THEOREM 2.1 *Let $0 < a \neq 1$. If $f\colon (-2, +\infty) \to \mathbb{R}$ satisfies the following conditions*

(i)

$$f(x) = a^{f(x-1)} \quad \textit{for all } x > -1, \quad f(0) = 1,$$

(ii) *f is differentiable on* $(-1,0)$ *and f' is a non-decreasing or non-increasing function on* $(-1,0)$,

(iii)

$$\lim_{x\to 0^+} f'(x) = (\ln a) \lim_{x\to 0^-} f'(x),$$

then f is uniquely determined through the equation

$$(*_3) \quad f(x) = \exp_a^{[x]}(a^{(x)}) = \exp_a^{[x+1]}((x)) \text{ for all } x > -2.$$

Proof First, note that the conditions (i) and (ii) imply that f is differentiable on (0,1). The condition (i) implies that

$$(*_4) \quad f(x) = \exp_a^{[x]}(f(x-[x])) \text{ for all } x > -2.$$

Put $\phi = f|_{(0,1)}$, considering (i) and (ii) [with due attention to $\lim_{x\to 0} f(x) = 1$], we have

$$\lim_{x\to 0^+} \frac{\phi'(x)}{\phi(x)} = \lim_{x\to 0^+} \phi'(x) = (\ln a) \lim_{x\to 0^-} f'(x) = \lim_{x\to 1^-} \frac{\phi'(x)}{\phi(x)}.$$

So, putting $\Phi(x) = \phi'(x)/\phi(x)$ $(0 < x < 1)$, we have Φ a is a non-decreasing or non-increasing function on (0, 1) and $\lim_{x\to 0^+} \Phi(x) = \lim_{x\to 1^-} \Phi(x)$, therefore, Φ is constant on (0, 1). Hence, $\phi(x) = a^x$ [note that $\lim_{x\to 1^-} \phi(x) = a$] and so $f(x) = a^x$ for all $x \in [0,1)$, therefore, we get $(*_3)$ by $(*_4)$. Conversely, if f is given by $(*_3)$, then f is defined on $(-2,+\infty)$ and clearly satisfies the conditions (i), (ii) and (iii). ■

COROLLARY 2.2 [A uniqueness conditions for $\exp_a$ on (0, 1)] *If g is a function defined on* (0, 1) *such that*

(i)

$$\lim_{x\to 0^+} g(x) = 1, \quad \lim_{x\to 1^-} g(x) = a, \quad \lim_{x\to 1^-} g'(x) = a \lim_{x\to 0^+} g'(x),$$

(ii) *the function $g'(x)/g(x)$ is non-decreasing or non-increasing on* (0, 1),

Then, $g(x) = \exp_a(x)$ for all $x \in (0,1)$.

COROLLARY 2.3 *The general solution of the equation $(*_2)$ can be written as*

$$(*_5) \quad f(x) = \exp_a^{[x]}(\varphi(x)),$$

where φ is any 1-periodic function. So, we have

$$(*_6) \quad f(x) = \exp_a^{[x]}(\phi((x))),$$

where ϕ is any function defined on $[0,1)$.

COROLLARY 2.4 *If f is a function defined on* $(-1, 1)$ *such that*

(i)

$$f(x) = a^{f(x-1)} \quad \textit{for all } x \in (0, 1), \quad f(0) = 1$$

(ii) f' *is a non-decreasing or non-increasing function on* $(-1, 0)$ *and*

$$\lim_{x\to 0^+} f'(x) = (\ln a) \lim_{x\to 0^-} f'(x),$$

then

$$f(x) = \begin{cases} 1 + x; & -1 < x \le 0 \\ a^x; & 0 \le x < 1. \end{cases}$$

[Note that (ii) implies $f'_+(0) = (\ln a) f'_-(0)$, so f is differentiable at 0 if and only if $a = e$].

Remark 2.5 Let f be the function satisfying the conditions of Theorem 2.1. If $a > 1$, then f cannot extend to $(-3, -2) \cup (-2, +\infty)$ by the ultra exponential functional equation [because $f(x) = \log_a(x + 2) < 0$ for $-2 < x < -1$]. But, if $0 < a < 1$, then it can be extended to $(-3, -2) \cup (-2, -\infty)$ [and, of course, the uniqueness theorem is valid for f: $(-3, +\infty) \setminus \{-2\} \to \mathbb{R}$ with considering (i) for all $-1 \neq x > -2$]. Next, if we consider this new function f, the domain f should be

$$\begin{aligned}(*_7)\quad D_f &= (-2, +\infty) \cup (-3, -2) \cup (-4 + a, -3) \cup (-5 + a, -5 + a^a) \\ &\quad \cup (-6 + a^{a^a}, -6 + a^a) \cup \cdots \\ &= \bigcup_{n=2}^{\infty} (-n - r(n)(a^{\underline{n-3}} - a^{\underline{n-4}}) + a^{\underline{n-3}}, -n + r(n)(a^{\underline{n-3}} - a^{\underline{n-4}}) + a^{\underline{n-4}}),\end{aligned}$$

where $r(n)$ is the remainder of the division of n by 2 and $a^{\underline{-2}}$ is taken to be $+\infty$. Now, if $x, x - 1 \in D_f$, then $f(x) = a^{f(x-1)}$ (it is possible that $x \in D_f$ and $x - 1 \notin D_f$ or $x + 1 \notin D_f$, for $x < -2$).

DEFINITION 2.6 *Fix* $0 < a \neq 1$. *We call the unique function satisfying the conditions of Theorem* 2.1 ultra exponential function *and denote it by* uxp_a *or* $\overline{\exp}_a$, *also we call* $\mathrm{uxp} = \mathrm{uxp}_e$ natural ultra exponential function. *Now, we can define ultra power by*

$$a^{\underline{x}} = \mathrm{uxp}_a(x) = \overline{\exp}_a(x).$$

Note that in view of $(*_7)$ *in Remark* 2.5, *if* $0 < a < 1$, *then we can extend the definition* $a^{\underline{x}}$ *to the larger domain* $(D_f = D_{\mathrm{uxp}_a})$.

Example $2^{\underline{4}} = 65536$, $e^{\underline{\pi/2}} = 5.868\ldots$, $0.5^{\underline{-4.3}} = 4.03335\ldots$, $0.6^{\underline{-5.264}} = -5.35997\ldots$, $0.7^{\underline{3.1}} = 0.7580\ldots$.

2.1 *Some properties of the ultra power and ultra exponential functions*

(i)

$$a^{\underline{x}} = a^{a^{\underline{x-1}}} \quad \text{for all } x > -1,$$

if $0 < a < 1$, then, for all $x, x - 1 \in D_{\mathrm{uxp}_a}$ ($=D_f$ in $*_7$), the above equation hold .

(ii) The function uxp_a is continuous on $(-2, +\infty)$ and differentiable on $(-2, +\infty) \setminus \mathbb{Z}$ and if $a > 1$, then uxp_a is increasing and

$$\lim_{x\to+\infty} a^{\underline{x}} = \begin{cases} +\infty; & a > e^{1/e} \\ \alpha; & \left(\frac{1}{e}\right)^e \le a \le e^{(1/e)} \\ \text{does not exist}; & 0 < a < \left(\frac{1}{e}\right)^e \end{cases}, \quad \lim_{x\to-2^+} a^{\underline{x}} = \begin{cases} -\infty; & a > 1 \\ +\infty; & a < 1, \end{cases}$$

where α is the least zero of the equation $a^x - x = 0$ [and also the limit of the sequence $\{a^{\underline{n}}\}$].

To explain, first of all note that if $a > 1$, then $a^{\underline{n}}$ is an increasing sequence and if $0 < a < 1$, then $a^{\underline{2n+1}}$ is an increasing and $a^{\underline{2n}}$ is a decreasing sequence. Also in refs. [1, 2], Euler shows that the sequence $a^{\underline{n}}$ is convergent (to α, where α is the least zero of f) if and only if $(1/e)^e \le a \le e^{1/e}$, and if $a > e^{1/e}$, then $\lim_{n\to+\infty} a^{\underline{n}} = +\infty$. Now, if $a > 1$, then $a^{\underline{[x]}} \le a^{\underline{x}} < a^{\underline{[x+1]}}$ and if $0 < a < 1$, then

$$(-1)^{[x+1]} a^{\underline{[x]}} \le (-1)^{[x+1]} a^{\underline{x}} < (-1)^{[x+1]} a^{\underline{[x+1]}}.$$

Therefore,

$$a^{\underline{[x]+r([x]+1)}} \le a^{\underline{x}} < a^{\underline{[x+1]-r([x]+1)}},$$

where $r([x] + 1)$ is the remainder of the division $[x] + 1$ by 2. Next, we get the result from the above inequalities.

(iii) The function $e^{\underline{x}}$ is continuously differentiable, but its second derivative at every integer point does not exist. It is convex on $[-1, +\infty)$ [uxp' is increasing on $[-1, +\infty)$]. The function uxp' satisfies the following functional equation (difference equation)

$$(*_8) \quad \chi(x) = e^{\underline{x}} \chi(x-1) \text{ for all } x > -1.$$

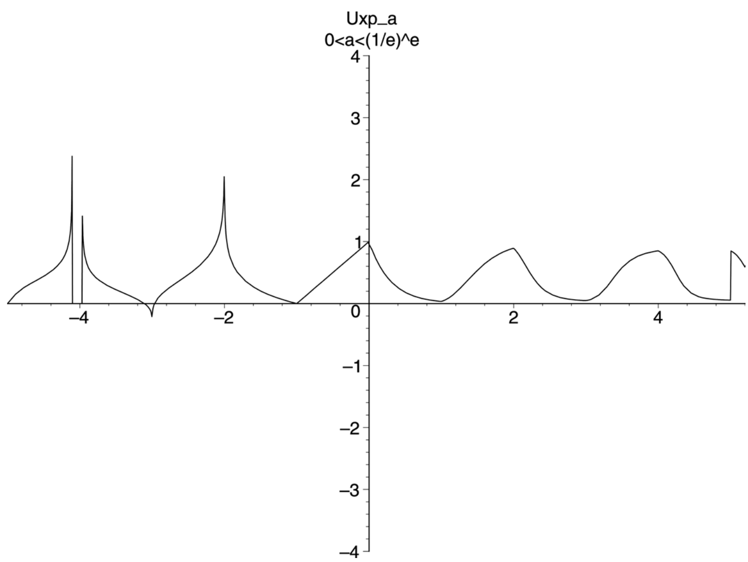

Figure 1. Divergently ultra exponential curves.

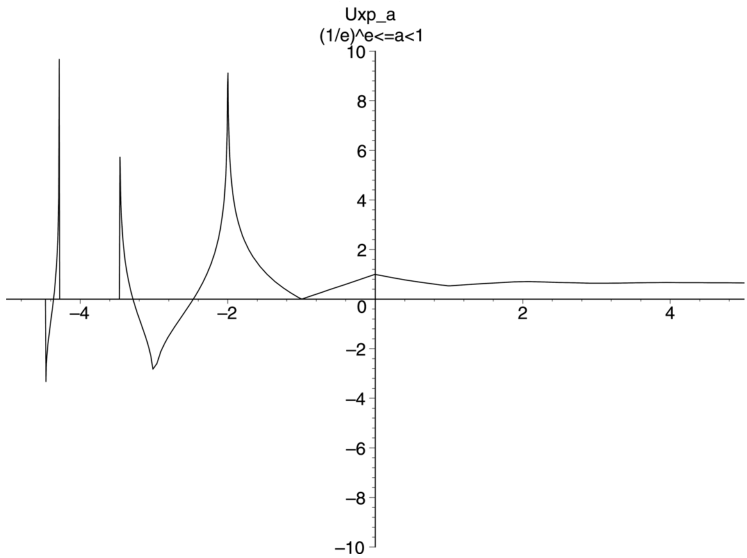

Figure 2. Convergently ultra exponential curves.

(iv) Therefore, we have five kinds of graph for the upper exponential functions, depended on range values of a (figures 1–5).

(v) The inverse function of uxp_a:
if $a > 1$, then uxp_a is invertible and

$$D_{\mathrm{uxp}_a^{-1}} = (-\infty, \alpha),$$

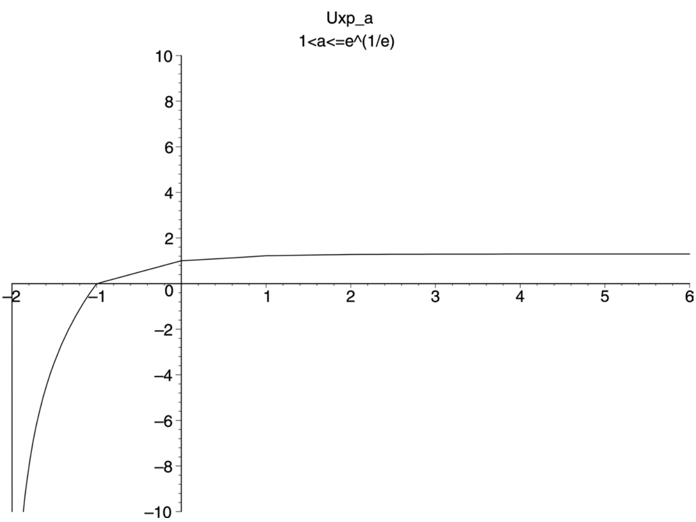

Figure 3. Increasingly convergently ultra exponential curves.

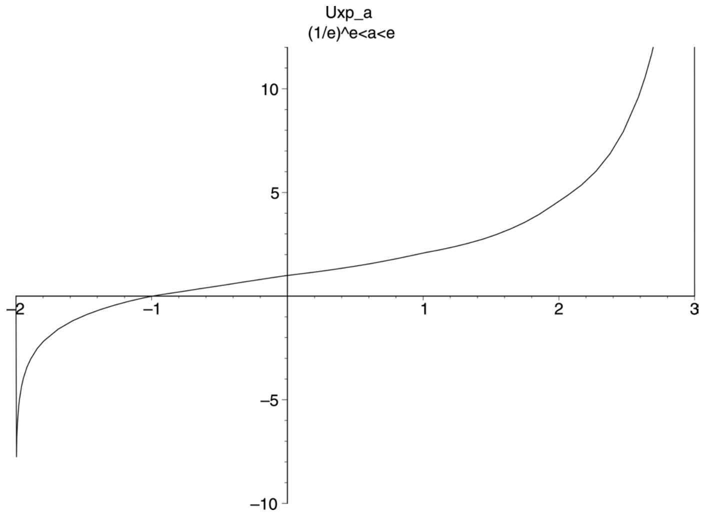

Figure 4. Unbounded ultra exponential curves.

where α is the limit of the sequence $\{a^{\underline{n}}\}$. Now, if $a > e^{1/e}$ for all x, and if $1 < a \le e^{1/e}$, for all $x < \alpha$, put

$$\overline{[x]}_a = \begin{cases} -2; & x < 0 \\ n; & a^{\underline{n}} < x < a^{\underline{n+1}}, \end{cases}$$

where $n \in \{-1, 0, 1, \cdots\}$. This definition is well defined, because for $a > 1$, uxp_a is continuous and increasing. Also, put

$$\overline{\log}_a(x) = \overline{[x]}_a + \log_a^{\overline{[x]}_a + 1}(x).$$

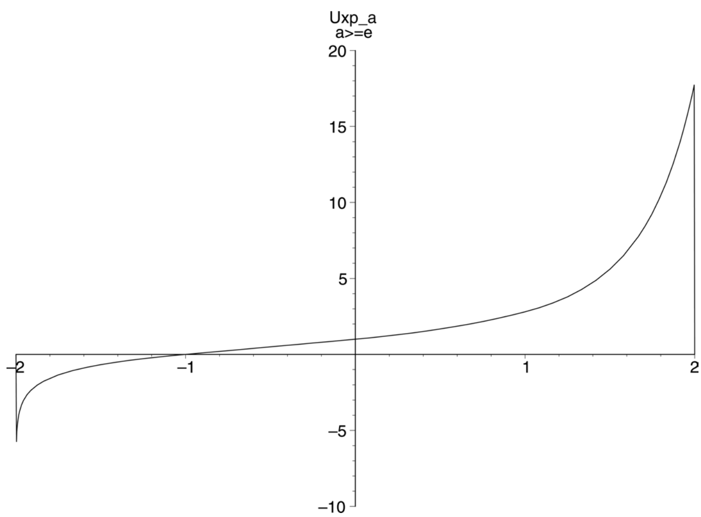

Figure 5. Convex ultra exponential curves.

Considering

$$\overline{[\mathrm{uxp}_a(x)]_a} = [x],$$

we have $\overline{\log}_a \, o \, \overline{\exp}_a = I_{(-2,+\infty)}$ and $\overline{\exp}_a \, o \, \overline{\log}_a = I_{(-\infty,\alpha)}$. Therefore,

$$\overline{\log}_a = \overline{\exp}_a^{-1}.$$

We call $\overline{\log}_a$ *infra logarithm function* (Iog_a).

3. Continuation of the uniqueness conditions to ultra exponential functions and an unsolved problem

Let I be an interval. A function f (defined on I) is called convex on I if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in I$ and $0 < \lambda < 1$. A function f is convex if and only if for every three elements $x, y, z \in I$ with $x < y < z$, one of the following inequalities hold

$$(*_9) \quad \frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(y)}{z-y},$$

$$(*_{10}) \quad \frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(x)}{z-x},$$

$$(*_{11}) \quad \frac{f(z)-f(x)}{z-x} \le \frac{f(z)-f(y)}{z-y}.$$

The inequality $(*_9)$ implies that if f is convex on open interval $I = (a, b)$, then one-sided derivatives of f ($f'_-(x)$, $f'_+(x)$) exist (on I) and are non-decreasing, *i.e.*

$$(*_{12}) \quad f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y),$$

for all $a < x < y < b$. Conversely, suppose the one-sided derivatives of f exist on $I = (a, b)$ and $(*_{12})$ holds and fixes $x, y, z \in (a, b)$ that $x < y < z$. Considering $(*_{12})$ the function $F(t) = f(t) - f(y) - f(x)/(y-x)(t-x)$ is continuous on $[x, y]$ and one-sided derivatives of it exist and non-decreasing. So [with due attention to $F(x) = F(y)$], the function F gets a relative minimum at $x < \xi_1 < y$ and so $F'_-(\xi_1) \le 0 \le F'_+(\xi_1)$, therefore,

$$f'_-(\xi_1) \le \frac{f(y)-f(x)}{y-x} \le f'_+(\xi_1).$$

Similarly, there is a $y < \xi_2 < z$ such that

$$f'_-(\xi_2) \le \frac{f(z)-f(y)}{z-y} \le f'_+(\xi_2).$$

But, $f'_+(\xi_1) \le f'_-(\xi_2)$, so

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(y)}{z-y}.$$

Therefore, f is convex on $I = (a, b)$.

If f is convex on (a, b) and f has left continuity at b, then f is convex on $(a, b]$ because if $a < x < y < b$, then choose the sequence b_n that $y < b_n \le b$ and $b_n \to b$ as $n \to \infty$. So,

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(b_n) - f(y)}{b_n - y}, \quad \text{for all } n.$$

Letting $n \to \infty$, we have $f(b_n) \to f(b)$ and

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(b) - f(y)}{b - y},$$

therefore, f is convex on $(a, b]$. Similarly, one can write for convexity on $[a, b)$ and $[a, b]$. If $a < c < b$ and f is convex on (a, c), (c, b) and $f'_-(c) \le f'_+(c)$, then f is convex on (a, b) (it is clear by $*_{12}$).

A positive function f is called log-convex if $\log f$ is convex.

LEMMA 3.1 *Let f be a convex function on $[a, b]$. If $f(x) > f(a)$ for all $a < x \le b$, then f is increasing on $[a, b]$.*

Proof Put

$$F(x) = \frac{f(x) - f(a)}{x - a} \quad a < x \le b,$$

F is non-decreasing on $(a, b]$. Now, if $f(x_2) \le f(x_1)$ for $a < x_1 < x_2$, then $F(x_1) > F(x_2)$ is impossible. Therefore, f is increasing on $[a, b]$. ■

THEOREM 3.2 *Assume $a > 1$ and f is a function defined on $[-1, 1)$ such that*

(i) $f(x) = a^{f(x-1)}$ *for all* $0 \le x < 1$, $f(0) = 1$,
(ii) f *is convex on* $(-1, 1)$, *then*
 (*a*) $a \ge e$ *and* $1 + x \le f(x) \le a^x$ *for all* $x \in [0, 1)$ [*and* $\log_a 2 + x \le f(x) \le 1 + x$ *for all* $x \in [-1, 0]$] *and*

$$\frac{1}{\ln a} \le f'_+(-1) \le 1 \le f'_-(0) \le f'_+(0) \le \ln a.$$

 (*b*) *If* $a = e$, *then* $f(x) = uxp(x)$, *for all* $x \in [-1, 1)$.
 (*c*) *If we extend* f *by the equation* $f(x) = a^{f(x-1)}$ *to* $(-2, +\infty)$, *then* f *is continuous, increasing on* $(-2, +\infty)$ *and convex on* $[-1, +\infty)$. *Furthermore*, f *is* log-*convex on* $[0, +\infty)$ *and we have since*

$$\lim_{x \to -2^+} f(x) = -\infty, \qquad \lim_{x \to +\infty} f(x) = +\infty.$$

Proof Fix $0 < x < 1$. f is continuous at 0, so $\lim_{x \to -1^+} f(x) = 0 = f(-1)$. Thus f is convex on $[-1, x]$ and so

$$\frac{f(x-1) - f(-1)}{x} \le f(0) - f(-1) \le \frac{f(x) - f(0)}{x}.$$

By (i) and $a > 1$, this inequality implies that $1 + x \le f(x) \le a^x$, $(1 + x)^{1/x} \le a$. So, $\log_a 2 + x \le f(x) \le 1 + x$ for all $x \in [-1, 0]$ and $a \ge e$. The above inequalities with $f(-1) = 0$, $f(0) = 1$ imply the last inequalities of (a). Now if $a = e$, then $f'_+(-1) = f'_-(0) = 1$, so $f'(x) = 0$ for all $-1 < x < 0$ and so $f(x) = \text{uxp}(x)$ for all $x \in [-1, 1)$. Suppose that f is the extension of the function f (by $f(x) = a^{f(x-1)}$) to $(-2, +\infty)$. Since f is convex on $(-1, 1)$, then f is continuous on $(-2, +\infty)$ and convex on $(n, n+1)$ for $n = -1, 0, 1, \ldots$ It is easy to see that

$$\frac{f'_+(n)}{f'_-(n)} = \frac{f'_+(0)}{f'_-(0)} \geq 1, \quad \text{for all } n \in \mathbb{N}^*,$$

so $f'_-(n) \leq f'_+(n)$, hence f is convex on $[-1, +\infty)$, and therefore, f is log-convex on $[0, +\infty)$ by the equality $f(x) = \log_a f(x+1)$ $(x \geq 0, a > 1)$. Since $f(x) > 0 = f(-1)$ for all $x > -1$ and f is convex on $[-1, +\infty)$, then f is increasing on $[-1, +\infty)$ by Lemma 3.1 and clearly on $(-2, +\infty)$. Finally, we have

$$\lim_{x \to -2^+} f(x) = \lim_{t \to -1^+} \log_a f(t) = -\infty,$$

and since $f(n) = a^{\underline{n}}$ for $n = -1, 0, 1, \ldots$ and f is increasing and $\lim_{n\to+\infty} a^{\underline{n}} = +\infty (a \geq e)$, then $\lim_{x\to+\infty} f(x) = +\infty$. ■

COROLLARY 3.3 *If f is a function defined on $(-1, 1)$ such that*

(i) $f(x) = e^{f(x-1)}$ *for all* $x \in (0, 1)$, $f(0) = 1$,
(ii) f *is convex on* $(-1, 0)$ *and* $f'_-(0) \leq f'_+(0)$ (*or if f is convex on* $(-1, \delta)$ *for some* $\delta > 0$),
then $f(x) = \text{uxp}(x)$ *for all* $x \in (-1, 0)$.

COROLLARY 3.4 *If $a \geq e$, then* uxp_a *is convex on* $[-1, +\infty)$ *and* log-*convex on* $[0, +\infty)$ *and if $a < e$, then* uxp_a *is not convex on* $[-1, 1)$ *or on* $[-1, +\infty)$.

COROLLARY 3.5 [A uniqueness conditions for the natural ultra exponential function.] *If* $f : (-2, +\infty) \to \mathbb{R}$ *is a function that*

(i) $f(x) = \exp(f(x-1))$ *for all* $x > -1$, $f(0) = 1$,
(ii) f *is convex on* $(-1, 0)$ *and* $f'_-(0) \leq f'_+(0)$,
then $f = \text{uxp}$.

Note: Comparing Corollary 3.3 to Corollary 2.4 and also Corollary 3.5 to Theorem 2.1 show that for $a = e$, the assumption of Corollary 2.4 and Theorem 2.1 (in case that we assume the function f' is nondecreasing) may be weakened, but still we get the same result.

Remark 3.6 Although for a $a > e$, we have shown in Theorem 3.2 that the graph of f is so closed to the function uxp_a, but we still could not show that completely $f = \text{uxp}_a$. So, we pose the following open question.

Unsolved Problem 3.7 Let $a > e$. Is there any convex function on $[-1, +\infty)$ except that $f = \text{uxp}_a$ which satisfies the ultra exponential functional equation?

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