In the paper "Ordered groups" there is given a survey of papers on ordered groups, reviewed in RZh Mathematika in 1975-1980.

Starting in 1963 there occurred a qualitative jump in the theory of ordered groups, evoked by the intensive investigation of linearly ordered (l.o.) groups and the development of the theory of groups of automorphisms of l.o. sets. As a result many sections of the theory of lattice ordered groups (l-groups) acquired an orderly and organized form, and profound classificational results were obtained in them. A natural consequence of this was the appearance in the recent past of several monographs on the theory of ordered groups, in particular, the books of Kokorin and Kopytov [26] (English translation [154]), Mura and Rhemtulla [182], Bigard, Keimel, and Wolfenstein [78].

The present survey is written on the materials of the Ref. Zh. "Matematika," mainly for the years 1975-1980, and reflects practically all directions of development of the theory of ordered groups with some accent on linearly and lattice ordered groups, which is explained by the intensivity and diversity of the investigations in these domains. In the survey there are included some results reviewed in the years 1970-1974 and not reflected in previous surveys of the collection "Algebra. Topology. Geometry" of the annual "Itogi Nauki i Tekhniki" [12, 13] or the book [26].

1. Linearly Ordered Groups

Investigations on l.o. groups were carried out mainly in the directions designated at the end of the sixties and formulated in [26]. The main progress in these directions was made for solvable groups.

Tests for Orderability and Description of Linear Orders on Groups. An element $x \neq e$ of the group $G$ is called generalized periodic if there exist $g_1, \ldots, g_n \in G$ such that $x = g_1^{-1}xg_1 \cdots g_n^{-1}xg_n \in G$. Bludov [4] constructed an example of an unorderable group without generalized periodic elements. An example of a solvable group with this property was constructed by Mura and Rhemtulla [179]. They also proved that a solvable group of finite rank without generalized periodic elements has linear orders. Kopytov found in [29] a test for orderability of solvable groups of finite rank in terms of irreducible integral polynomials. Bludov [204] proved that if a group can be approximated by finite p-groups for an infinite set of primes $p$, then it admits linear orders. This result was improved by Kopytov and Medvedev [33]. Namely, such a group admits a linear order in which the system of its convex subgroups is central. The latter result allowed them to get by one method a series of facts known earlier about groups, approximable by finite p-groups.

Bludov [6] gave a new example of a group admitting a unique linear order. Kopytov [29] proved that if the number of ordered solvable groups is finite, then it is a multiple of 4. For any $k > 1$ there exists a solvable group, admitting precisely $4k$ linear orders. Pilz [191] characterized all linear orders of the direct product of linearly ordered groups.

Systems of Convex Subgroups in L. O. Groups. Medvedev studied in [38] orderable groups which have only a finite number of relatively convex groups. Each such finitely generated group has nilpotent commutator subgroup, whose quotient-group with respect to the isolator is infinite cyclic, and the rank of such a group is finite. Mura and Rhemtulla [180] proved that each orderable group with a finite number of relatively convex subgroups and each of its quotient-groups being a torsion-free group, is hyperorderable. Kopytov [29] proved that each solvable l.o. group has a proper normal convex subgroup. He constructs an example of a locally solvable O-simple 1.o. group.

It is well known that in l.o. groups there holds identically the inequality $|[x, y]| \leq |x| \cdot |y|$, and the inequality $|[x, y]| \leq |x|$ does not always hold. Kopytov and Medvedev [33] studied l.o. groups for which the latter

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inequality holds identically. Such groups are called rigid I.o. groups. A I.o. group is rigid if and only if the system of its convex subgroups is central. In the language of normal subgroups conditions are found for the existence on a group of a rigid linear order.

Imbedding and Extension. Bludov and Medvedev [9] proved that any orderable metabelian group can be imbedded in a metabelian orderable group in which the equation $x^n = a$ is solvable for any element $a$. Fox in [108, 109] proved that for any Abelian normal subgroup $A$ of the I.o. group $G$ there exists an ordered imbedding $\varphi$ of the I.o. group $G$ in an I.o. group $G^*$ such that the equation $x^n = \varphi(a)$ is solvable in $G^*$ for all $a \in A$ and integral $n$. If $G$ is hyperorderable, then $G^*$ is also hyperorderable. In the supplement [109] Fox established that if $G$ is contained in the manifold $\mathfrak{M}$ of groups, then $G^*$ can also be chosen in $\mathfrak{M}$.

Elliott [104] showed that any countable Abelian I.o. group is an inductive limit of a sequence $Z^{r_1} \varphi_1 \ldots \varphi_{r_2} Z^{r_2} \varphi_2 \ldots$ of direct products $Z^{r_1}$ of cyclic I.o. groups with coordinatewise order, while all the maps $\varphi_i$ are ordered imbeddings.

Mura and Rhemtulla proved in [183] that if $A$ is a normal subgroup of a free nonabelian group $F$, $V$ is a completely characteristic subgroup of $A$ such that $A/V$ is orderable, $F/A$ is right orderable and has an invariant system of subgroups with factors from the manifold of groups generated by $A/V$, then $F/V$ is orderable. Chehata and Wiegandt [93] constructed a theory of radicals in I.o. groups, analogous to the Kurosh-Amitsur theory for rings. They introduced the concepts of radical and semisimple class of I.o. groups and studied properties of such classes. The class $J$ is radical if and only if $J$ is closed with respect to $O$-homomorphisms and transfinite extensions. The class $I$ is semisimple if $I$ is closed with respect to taking normal convex subgroups and transfinite coextensions. There exist radical classes which are simultaneously semisimple.

Properties of L. O. Sets of L. O. Groups. Koppelberg [156] proved that a l.o. group, whose l.o. set is complete and any set of pairwise disjoint nonempty intervals is countable, is separable. Ermolov [19] established that for any countable l.o. group $G$ and any of its normal subgroups $H$ one can find a countable Abelian group $G^*$, a subgroup $H^*$ of it, and a one-for-one map $\varphi: G \to G^*$ such that $\varphi(H) = H^*$ and $\varphi$ induces a map of $G/H$ onto $G^*/H^*$, where the systems of convex subgroups of $G$ and $G^*$ have the same order type. Goffman [121] analyzed the relation between the concepts of order and topological completeness on the class of l.o. groups. In [155] there were considered elementary properties of Abelian I.o. groups, and the decidability of the theories of certain concrete classes of Abelian l.o. groups was proved.

Right Ordered Groups. Mura and Rhemtulla [181] considered the following subclasses of the class RO of right orderable groups: $C_1$ consists of groups whose system of convex subgroups is solvable for any right order; $C_2$ consists of groups any subgroup of which is a $C_1$-group; $C_3$ consists of groups $G$ such that for any two elements $x, y$ one can find $u, v$ from the subgroup generated by the elements $x, y$, such that $ux = vy$. Then $RO \supset C_1 \supset C_2 \supset C_3 \cap RO$, while all the inclusions are strict. Any locally solvable $C_2$-group is locally an extension of a nilpotent group by a finite group. A finitely generated solvable orderable $C_3$-group is nilpotent.

Mura [177] constructed an example of a polycyclic right-ordered group, whose system of convex subgroups is not subnormal. Ault [68, 69] proved that for a locally nilpotent right-ordered group the system of convex subgroups is solvable. Rhemtulla [203] established that if the group $G$ is finitely generated, the quotient group by its commutator subgroup $G'$ is finite and $G'$ has nilpotent subgroup of finite index, then $G$ does not admit right orders. Each partial right order of a torsion-free nilpotent group extends to a linear right order. An analogous result was also obtained by Formanek [107]. Solvable groups for which each partial right order extends to a linear right order were studied by Pierce [189].

2. Lattice Ordered Groups

Properties of Lattices of l-Groups. Completely distributive l-groups were studied by Jakubik [135]. For cardinal numbers $\alpha, \beta$ the l-group $G$ is called $(\alpha, \beta)$-distributive if for any sets of indices $T, S$ such that $|T| \leq \alpha, |S| \leq \beta$ the equation $\bigwedge_{t \in T} \bigvee_{s \in S} x_{ts} = \bigvee_{q \in S} \bigwedge_{t \in T} x_{qt}$ holds for any sets $\{x_{ts} | t \in T, s \in S, x_{ts} \in G\}$, for which at least one of the terms of this equation is defined. It is proved that any $(\alpha\beta)$-distributive l-group is $(\alpha, \alpha)$-distributive. Redfild [199, 200] introduced the concept of Archimedean element, i.e., an element $a$ of the l-group $G$, such that $a > e$, and for any $g \in G$, $e < g < a$, one can find a natural number $n$, for which $g^n < a$. The subgroup $A(G)$ of the l-group $G$, generated by all Archimedean elements of $G$, is the largest Archimedean convex l-subgroup of $G$. If $G$ is an l-group in which each element exceeds some Archimedean element, then $G$ is completely distributive if and only if it has a basis.

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An element \( s \) of the \( l \)-group \( G \) is called singular if \( s \preceq e \) and for any decomposition \( s = uv \), where \( u \preceq e \), \( v \preceq e \), one has \( u \wedge v = e \). Lattice ordered groups in which each positive element exceeds some singular one, are called singular. Wolfenstein [222, 223] characterized singular complete, singular Archimedean, and singular hyperarchimedean (i.e., those \( l \)-groups each \( l \)-homomorphic image of which is Archimedean) \( l \)-groups. This characterization is given in terms of representations of \( l \)-groups by continuous functions, defined on a suitable locally compact completely disconnected set with values in the topological completion \( Z \) of the \( l \)-group \( Z \) of integers in the topology induced by the order. Martinez [163] proved that any sequence \( a_1 > \ldots > a_n > \ldots \) of elements of the \( l \)-group \( G \) such that \( a_n \geq a_{n+1} \), terminates if and only if \( G \) is the union of an increasing sequence of \( l \)-ideals \( G_{\alpha} \) where for any \( \alpha \), \( G_{\alpha+1}/G_{\alpha} \) is generated by its singular elements. If in addition \( G \) is hyperarchimedean, then any \( l \)-ideal of \( G \) is cyclic.

Berman [73] proved that for any \( l \)-group \( G \) the semigroup of automorphisms \( f \) of the lattice \( G \), for which \( f(x) \geq x \) for any \( x \in G \), is transitive. Franchello [112] established that the sublattice \( L \) of the \( l \)-free product of \( l \)-groups \( G_{\alpha} \), generated by the free factors, is the free product (in the category of distributive lattices with distinguished element \( e \)) of the lattices \( G_{\alpha} \).

Systems of Subgroups in \( l \)-Groups. Jakubikova proved in [146] that the lattice of all \( l \)-subgroups of the \( l \)-group \( G \) is distributive if and only if \( G \) is an Abelian \( l \).o. group of rank 1.

One says that the \( l \)-group \( G \) has the splitting property if \( G \) is distinguished as an \( l \)-direct factor in any \( l \)-group \( H \), containing \( G \) as an \( l \)-ideal. Jakubik [131] studied the splitting property for Archimedean, complete, singular Archimedean, and for orthocomplete \( l \)-groups. Andersen, Conrad, and Kenny [67] considered the connection between the splitting property and essential extensions of \( l \)-groups. A lattice ordered group \( H \), \( G \in H \), is called an essential extension of the \( l \)-group \( G \) if for each convex \( l \)-subgroup \( A \), \( A \neq E \), of the \( l \)-group \( H \) one has \( G/\Gamma A \neq E \). An \( l \)-group which does not have proper essential extensions is called an essentially closed \( l \)-group. Earlier Conrad proved that any Archimedean \( l \)-group has a unique up to \( l \)-isomorphism essential closure. In [67] it is proved that an Archimedean \( l \)-group \( G \) has the splitting property if and only if \( G \) coincides with any of its essential Archimedean extensions in which it is an \( l \)-ideal. Any essentially closed Archimedean \( l \)-group has the splitting property.

We recall that by the polar \( X^\perp \) of the set \( X \) of the \( l \)-group \( G \) is meant the set of all elements of \( G \), orthogonal to each element of \( X \). Any polar of an \( l \)-group is a convex \( l \)-subgroup of it. A lattice ordered group \( G \) is called a P-group (SP-group), if each of its polars (the polar of any element \( g \neq e \)) is an \( l \)-direct factor of \( G \). If \( H \) is an essential extension of the \( l \)-group \( G \) and \( H \) is a P-group (SP-group), where no proper \( l \)-subgroup of \( H \) containing \( G \) is a P-group (SP-group), then \( H \) is called the P-hull (SP-hull) of \( G \). We recall that the \( l \)-group \( G \) is called O-approximable (or representable), if \( G \) is isomorphic with an \( l \)-ideal of \( G \). Wolfenstein [222, 223] characterized singular complete, singular Archimedean, and singular hyperarchimedean (i.e., those \( l \)-groups each \( l \)-homomorphic image of which is Archimedean) \( l \)-groups. This characterization is given in terms of representations of \( l \)-groups by continuous functions, defined on a suitable locally compact completely disconnected set with values in the topological completion \( Z \) of the \( l \)-group \( Z \) of integers in the topology induced by the order. Martinez [163] proved that any sequence \( a_1 > \ldots > a_n > \ldots \) of elements of the \( l \)-group \( G \) such that \( a_n \geq a_{n+1} \), terminates if and only if \( G \) is the union of an increasing sequence of \( l \)-ideals \( G_{\alpha} \) where for any \( \alpha \), \( G_{\alpha+1}/G_{\alpha} \) is generated by its singular elements. If in addition \( G \) is hyperarchimedean, then any \( l \)-ideal of \( G \) is cyclic.

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Fleischer [106] considered decomposition into a Cartesian $l$-product of $l$-groups with operators. Miller [173] studied $l$-direct factors in $l$-groups. Jakubikova [147] considered $l$-groups $G$, in which each $l$-direct factor is isomorphic with $G$, and $l$-groups in which this property fails. Conrad [96] proved that a vector lattice $G$ of countable dimension over an l.o. field under certain assumptions can be approximated by order simple l.o. spaces. In [98] there were considered essential extensions and automorphisms of vector lattices.

Orthogonally Complete $l$-Groups. Various versions of the concept of orthogonally complete were considered in several papers. The most natural definition of an orthogonally complete $l$-group was given by Bernau first for P-groups, and later extended to arbitrary $l$-groups. Reducing the account to a unified terminology, we shall call an $l$-group orthocomplete if each set of positive pairwise orthogonal elements of $G$ has in $G$ a least upper bound. Bernau [74, 75] proved that for any $l$-group $G$ there exist a unique up to $l$-isomorphism orthocomplete $l$-group $H$ and $l$-isomorphism $\varphi: G \to H$, such that $\varphi(G)$ is dense in $H$ and between $\varphi(G)$ and $H$ there are no orthocomplete $l$-subgroups. In this case $H$ is called the orthocompletion of $G$. For $O$-approximable $l$-groups, Bleier [80] gave another construction of the orthocompletion. Bernau [74] proved that for Archimedean $l$-groups the operations of orthocompletion and Dedekind completion are interchangeable. Jakubik [132] established that if $G$ is orthocomplete and any of its countable bounded subsets has a least upper bound in $G$, then $G$ is complete, i.e., any bounded subset of $G$ has a least upper bound in $G$. An analogous result was obtained by Rotkovich [54]. In [55] it is proved that in Archimedean $l$-groups, from the existence in $G$ of a least upper bound of any pair of orthogonal positive elements follows the orthocompleteness of $G$. For other connections between the concepts of completeness, conditional completeness, orthocompleteness, and Archimedeanness, cf. [53, 56, 141].

Complete $l$-Groups. Imbeddings. It is well known that any complete $l$-group is Archimedean. A subgroup $H$ of the $l$-group $G$ is called closed if for any subset $M$ of $H$ such that in $G$ there exists a least upper bound $\sup G M$, there also exists $\sup H M$, while $\sup G M = \sup H M$. Jakubikova [148, 149] introduced the concept of a generating set in a complete $l$-group $G$, i.e., a subset $X$ such that the closed convex $l$-subgroup of $G$ generated by the set $X$ coincides with $G$. She studied questions of existence and properties of free objects with two generators in the classes of complete, complete singular, complete without singular elements, complete orthocomplete, and complete completely distributive $l$-groups.

In [113] under certain assumptions there was given a decomposition of a complete $l$-group into a product of convex $l$-subgroups of special form.

Jakubik [132, 140] generalized the concept of Dedekind completion $D(A)$, familiar for Archimedean $l$-groups $A$, to the case of an arbitrary $l$-group. He proved the existence in any $l$-group $G$ of a largest Archimedean convex $l$-subgroup $A(G)$, which he calls the Archimedean kernel of $G$. By the generalized Dedekind completion $D_l(G)$ of the $l$-group $G$ is meant an $l$-group such that: 1) $G$ is an $l$-subgroup of $D_l(G)$; 2) $D(A(G))$ is an $l$-ideal in $D_l(G)$; 3) if $x \in G$ and $X$ is a nonempty subset of $x A(G)$, bounded in $x A(G)$, then there exists $x_0 \in D_l(G)$ such that $x_0 = \sup X$; 4) for any $x_0 \in D_l(G)$ there exists $x \in G$ and $X \subset x A(G)$ such that $x_0 = \sup X$. The existence and uniqueness of $D_l(G)$ is proved for an arbitrary $l$-group $G$. If $A(G)$ is closed in $G$, then $D(A(G))$ is closed in $D_l(G)$. Let $A$ be any of the classes of $l$-groups: Archimedean, $O$-approximable, Abelian divisible. If $G \in A$, then $D_l(G) \in A$. In [142] it is established that $D_l(i G) = i D_l(G)$. In [141] it is proved that the classes of $SP$-groups and conditionally orthocomplete $l$-groups are closed with respect to taking generalized Dedekind completions. The connection between higher degrees of distributivity for $G$ and $D_l(G)$ is studied. In [139, 142] Jakubik introduced and studied the concept of maximal Dedekind completion and compared it with the concept of generalized Dedekind completion. Koldunov [27] considered the connection between Dedekind and $O$-completions of Archimedean $l$-groups.

If in the $l$-group $G$ the $l$-subgroup $H$ is such that the map $X \to X \cap H$ is a one-to-one mapping of the set of convex $l$-subgroups of $G$ onto the set of closed (in $H$) convex $l$-subgroups of $H$, then $G$ is called an $a^*$-extension of $H$. $G$ is called $a^*$-closed if it does not have proper $a^*$-extensions.

In [119] the existence was proved of $a^*$-closures for completely distributive, and in [70] for arbitrary $l$-groups. $a^*$-closures of $l$-groups were also studied in [82]. Bludov [5] constructed an example of an Archimedean complete (i.e., not having properly linearly ordered essential extensions) l.o. group, not all the factors of jumps of convex subgroups of which are isomorphic with the additive group of real numbers in the natural order. Essential extensions of $l$-groups were also studied by Kenny [151], Loonstra [159], Conrad [94] considered imbeddings of an Abelian $l$-group $G$ in the group $V(D)$ of all real-valued functions, defined on the root system $A$, and such that for any $f \in V(D)$ the set of those $a \in A$ such that $f(a) = 0$, satisfies the condition of maximality. Other types of imbeddings, similar to those listed, were considered in [209, 90, 11, 187].
By an amalgam $\langle G_1, G_2, H, \alpha_1, \alpha_2 \rangle$ is meant a collection of $l$-groups $G_1, G_2, H,$ and $l$-isomorphisms $\alpha_i: H \to G_i$ ($i = 1, 2$). One says that the class $\mathfrak{S}$ of $l$-groups has the property of amalgamability, if for any amalgam $\langle G_1, G_2, H, \alpha_1, \alpha_2 \rangle$, $G_1, G_2 \in \mathfrak{S}$, there exists an $l$-group $K \in \mathfrak{S}$ and $l$-isomorphisms $\phi_i: G_i \to K$ such that $\alpha_1 \phi_1 = \alpha_2 \phi_2$. Pierce [188, 190] proved that the class of all $l$-groups does not have the property of amalgamability, and gave some classes of $l$-groups, having this property. With the help of imbedding an amalgam of $l$-groups in an $l$-group, he proved that any $l$-group can be imbedded in a divisible $l$-group and in an $l$-group in which any two positive elements are conjugate. Glass [116] considered manifolds $\mathfrak{M}$ of $l$-groups with the property HNN: for any $l$-group $G \in \mathfrak{M}$ and any pair of isomorphic $l$-subgroups $A, B$ in $G$, one can find an $l$-group $H \in \mathfrak{M}$, containing $G$ as an $l$-subgroup and such that $A$ and $B$ are conjugate in $H$. It turns out that any amalgam of $l$-groups from a manifold $\mathfrak{M}$ of $l$-groups with the property HNN can be imbedded in an $l$-group from $\mathfrak{M}$.

Manifolds of $l$-Groups. In correspondence with the general theory of algebraic systems, the class $\mathfrak{S}$ of all $l$-groups is a manifold of signature $l = \langle \cdot, 1, \cdot, \cdot, \vee, \wedge \rangle$. By a manifold of $l$-groups (an $l$-manifold) is meant any submanifold of the manifold $\mathfrak{S}$, i.e., any class consisting of all $l$-groups satisfying some collection of identities of signature $l$. Although isolated results on $l$-manifolds appeared even earlier, the intensive study of identical relations in $l$-groups began about 1975. Martinez [160] considered a series of questions of the general theory of manifolds of $l$-groups, established the simplest properties of operations on the set of $l$-manifolds and similar classes of $l$-groups. Bernau [76] proved the closedness of each $l$-manifold with respect to orthocompletion.

A very important role in the theory of $l$-manifolds is played by the class $\mathfrak{S}$ of all $l$-groups with subnormal jumps. Wolfenstein [221] proved that $\mathfrak{S}$ is defined in $\mathfrak{S}$ by the identity $1 \cdot 1 \cdot 1 = 1 \cdot 1 \cdot 1$. Already in 1959 P. G. Kontorovich and K. M. Kutyev established that the class of $O$-approximable $l$-groups is also an $l$-manifold and is defined by the identity $(x \wedge y) \vee (x \wedge y) = x \vee y$. It was also known long ago that the class of all $l$-groups $G$ for which any convex $l$-subgroup is an $l$-ideal and for any jump $A < B$ of $l$-ideals one has $B \subseteq A_1$, is defined by the identity $1 \cdot 1 \cdot 1 = 1 \cdot 1 \cdot 1$. Such $l$-groups we call rigid $l$-groups. Kopytov [30] proved that any locally nilpotent $l$-group is $O$-approximable, and moreover a rigid $l$-group. Any lattice order of a locally nilpotent group is the intersection of linear orders.

In [28, 31] there is given a method of construction of rigid $l$-groups from ordered Lie algebras.

Lattice of $l$-Manifolds. The set of all $l$-manifolds is a complete distributive lattice with respect to the natural operations of union and intersection, which follows immediately from the results of G. Birkhoff in the 1940s. Kopytov and Medvedev [34] established that the lattice of all $l$-manifolds has the cardinality of the continuum and is not completely distributive.

In [125] it was proved that the $l$-manifold $\mathfrak{S}$ of $l$-groups with subnormal jumps is the largest proper submanifold of the manifold of all $l$-groups. In [208] there was given an infinite series of $l$-manifolds $\mathfrak{S}_e$, covering the smallest $l$-manifold $\mathfrak{S}_e$, of all Abelian $l$-groups. None of the manifolds $\mathfrak{S}_e$, lies in the manifold $\mathfrak{S}$, of $O$-approximable $l$-groups. Medvedev [39] constructed three more $l$-manifolds $\mathfrak{S}_i$ ($i = 1, 2, 3$), covering the manifold $\mathfrak{S}_e$, and proved that any $l$-manifold $\mathfrak{S}_e$, containing at least one solvable nonabelian i.o. group, contains one of the manifolds $\mathfrak{S}_i$ ($i = 1, 2, 3$). In [34] it is proved that there exists an $l$-manifold, not generated by a finitely generated $l$-group.

Free $l$-Groups. The study of free $l$-groups, started at the end of the 1960s, continued in several directions. Kopytov [32] got the following description of free $l$-groups of arbitrary $l$-manifolds, analogous to the Conrad-Weinberg description of free Abelian $l$-groups and the Conrad description of $l$-groups, free in $\mathfrak{S}_e$. Let $\mathfrak{S}_i$ be an arbitrary $l$-manifold, $\mathfrak{S}(\mathfrak{S}_i)$ be the class of groups, imbeddable as subgroups in an $l$-group from $\mathfrak{S}_i$. It turns out that $\mathfrak{S}(\mathfrak{S}_i)$ is a quasimanifold of groups. Let $F_0$ be a free group of $\mathfrak{S}(\mathfrak{S}_i)$ with basis $X = \{x_1, x_2, \ldots \}$. By $F_0^\alpha$ we denote the group $F_0$ provided with the right order $F_0^\alpha$, and by $N_0^\alpha$ the smallest convex subgroup of $F_0^\alpha$ such that $A_0 \subseteq \mathfrak{S}_i$, where $A_0$ is the $l$-group generated by the right regular representation of $F_0^\alpha$ by automorphisms of the i.o. set $\mathcal{R}_{K_0}(F_0^\alpha)$ of right cosets of $F_0^\alpha$ by $N_0^\alpha$. By $F$ we denote the Cartesian $l$-product of all $A_0$, when $F_0^\alpha$ runs through the set of all right orders of $F_0$. Let $F$ be the $l$-subgroup of $F$, generated by the elements $x_i: x_i(x_0) = R_0(x_i)$, where $R_0(x_i)$ is the right translation of the set $\mathcal{R}_{K_0}(F_0^\alpha)$, induced by the element $x_i$, i.e., $R_0(x_i)(N_0^\alpha y) = N_0^\alpha x_i y$. It is proved that $F$ is an $l$-group which is free in $\mathfrak{S}_i$, $X = \{x_1, x_2, \ldots \}$ is a free basis of $F$. If $F_0$ is a free group of countable rank, then there exists a right order $\preceq$ on $F_0$ such that the $l$-subgroup of the $l$-group of automorphisms of the $l$-o. set $\langle F_0, \preceq \rangle$, generated by the right translations of $F_0$, is a free $l$-group. There is described in detail the analogous representation of free $\mathfrak{S}_i$, also in the $l$-manifold of rigid $l$-groups.
In [81] it is proved that a free Abelian \( l \)-group is characteristically simple. A free \( l \)-group cannot contain an uncountable set of pairwise orthogonal elements. Free Abelian \( l \)-groups over distributive lattices were considered by Arkhipova [2, 3].

**Torsion Classes. Radical Classes.** Martinez [162] called a torsion class a class \( \mathfrak{F} \) of \( l \)-groups, closed with respect to \( l \)-homomorphisms, taking \( l \)-subgroups, and such that in any \( l \)-group \( G \) there is contained a convex \( l \)-subgroup \( \mathfrak{F}(G) \), belonging to \( \mathfrak{F} \) and containing all convex \( l \)-subgroups of \( G \) from the class \( \mathfrak{F} \). \( \mathfrak{F}(G) \) is called the \( \mathfrak{F} \)-radical of \( G \). The question of the possibility of reconstructing the torsion class from the \( \mathfrak{F} \)-radical was solved positively by Jakubik [138]. The set of all torsion classes forms a complete Brauer lattice (Martinez [164]). A torsion class is called complete is it is closed with respect to extensions. Holland and Martinez [128] proved that complete and certain other torsion classes are join-irreducible.

Holland [127] proved that any \( l \)-manifold is a torsion class. The following classes are torsion classes: hyperarchimedean \( l \)-groups [161, 95], the classes of all Cartesian \( l \)-products of \( l \)-groups, Abelian \( l \)-groups, \( l \)-groups of integers. A large number of other torsion classes and relations among them are given in Conrad's survey [99] and in [134, 144, 165].

The class of Archimedean \( l \)-groups is not closed with respect to \( l \)-homomorphisms and hence is not a torsion class. Nevertheless, in any \( l \)-group \( G \) there exists an Archimedean kernel \( A(G) \), containing all Archimedean convex \( l \)-subgroups of \( G \). Properties of \( A(G) \) were studied in [81, 91, 92, 139, 140, 142, 152].

**Algorithmic Questions in \( l \)-Groups.** Although the class of all \( l \)-groups is a manifold of signature 1, it turns out that the class of all groups, which admit a lattice order, nonaxiomatizable in the signature of the group (Vinogradov [14]).

Holland and McCleary proved in [129] that the problem of equality of words in a free \( l \)-group is algorithmically unsolvable. Glass [115] constructed an example of a recursively representable \( l \)-group with unsolvable word problem. Holland [126] considered the question of the equality to the identity of certain types of group words in a free product of \( l \)-groups. Jakubik [143] gave a new method for defining \( l \)-groups in terms of formal operations of signature 1. Kutyev in [35, 36, 37] characterized \( l \)-groups in terms of the lattice of subsemigroups.

**Isometries in \( l \)-Groups.** In [215, 216] there is introduced the concept of an isometry \( f \) in an Abelian \( l \)-group \( G \) as a mapping of \( G \) onto \( G \) such that for all \( x, y \in G \) one has \( f(x) - f(y) = |x - y| \). It was shown that for any isometry \( f \) of an Abelian \( l \)-group \( G \) one can find an isometric involutive automorphism \( T \) of the \( l \)-group \( G \) such that for any \( x \in G \) one has \( f(x) = T(x) + f(0) \). In particular, the isometry \( f \) preserves order if and only if \( f \) is a translation of \( G \): \( f(x) = x + f(0) \). By \( G^* \) one denotes the group of all one-to-one isometries of \( G \) with the order \( f \) if and only if \( f(x) \leq g(x) \) for any \( x \in G \). Any \( l \)-isomorphism of \( G \) onto \( H \) induces an order isomorphism of \( G^* \) onto \( H^* \), carrying translations into translations.

Jakubik [145] extended the concept of isometry to an arbitrary \( l \)-group \( G \), considering a one-to-one mapping \( f \) of \( G \) onto an isometry \( f \) of \( G \) an isometry if for any \( x, y \in G \) one has \( f(x)f(y)^{-1} = |x - y| \). It was shown that each isometry is a combination of an \( e \)-isometry and a translation. For any \( e \)-isometry \( f \) there exists a unique decomposition of \( G \) into a direct \( l \)-product \( G = A \times B \), such that \( f(x) = x(A)x(B)^{-1} \), where \( x(A) \), \( x(B) \) are the projections of \( x \) onto \( A, B \), respectively.

In [77] the theory of duality is extended to finitely generated Abelian \( l \)-groups and projective objects in the category of Abelian \( l \)-groups are studied.

**3. Groups of Automorphisms of Linearly Ordered Sets**

It is well known that the group \( \text{Aut}(X) \) of all automorphisms of the \( l \)-set \( X \) is an \( l \)-group with respect to the following order: \( f \leq g \) in \( \text{Aut}(X) \) if and only if \( f(x) \leq g(x) \) for all \( x \in X \). Holland's theorem (1962) that any \( l \)-group has a faithful representation by automorphisms of a \( l \)-set emerged as a powerful instrument for the study of \( l \)-groups and gave impetus to the intensive investigation of groups of automorphisms of \( l \)-sets. The basic features of the theory of automorphism groups were developed at the end of the 1960s and the start of the 1970s, and cannot be treated completely in the present survey. Here we shall give only the main concluding results and their applications to certain questions of the theory of \( l \)-groups.

Most of the concepts of the theory of representations of groups by permutations can be interpreted naturally in the group of automorphisms of a \( l \)-set. We shall give some of these concepts which are needed in what follows. By the stabilizer \( \text{St}_G(a) \) of the point \( a \) of the \( l \)-set \( X \) in the group of automorphisms \( G \leq \text{Aut}(X) \) is meant the set of those \( g \in G \) such that \( g(a) = a \). By \( X(a, x) \) for \( a, x \in X \) we denote the set \( \{ g(a) | g \in \text{St}_G(a) \} \). The
group $G$ of automorphisms of the l.o. set $X$ is called periodic if there exists an automorphism $t$ of the Dede-
kind closure $X$ of the l.o. set $X$, commuting elementwise with $G$ on $X$ such that $t$ has no fixed points in $X$, and
for any $a \in X$ one has $\{y \in X | a < y < t(a)\} = X(a, x)$, where $x$ is any element of $X$, $a < x < t(a)$.

O-Primitive Groups. The group $G$ of automorphisms of the l.o. set $X$ is called O-primitive if on $X$
does not exist a nontrivial convex equivalence relation $\theta$ such that $x \equiv y \pmod{\theta}$ implies $g(x) \equiv g(y) \pmod{\theta}$
for all $x, y \in X$. O-primitive groups are analogs of simple groups and are the "bricks" out of which all groups
of automorphisms of l.o. sets are built.

In [168, 169] there is concluded a cycle of papers on transitive O-primitive l-groups of automorphisms
of l.o. sets and the following classification theorem is proved. If $G$ is a transitive O-primitive l-group of
automorphisms of the l.o. set $X$, then $G$ is a group of one of the following types: 1) $G$ is an Archimedean l.o.
group; 2) $G$ is O-2-transitive on $X$, i.e., for any $x_1, x_2, y_1, y_2 \in X$, $x_1 < y_1, x_2 < y_2$, there exists a $g \in G$
such that $g(x_1) = x_2, g(y_1) = y_2$; 3) $G$ is a periodic group of automorphisms of $X$, while the stabilizer of each point $a$
of $X$ in $G$ acts faithfully on each set $X(a, x)$ and is an O-2-transitive group of automorphisms of $X(a, x)$, having
an element with carrier bounded in $X(a, x)$.

In [72] there was studied the correspondence between right-regular representations of right-ordered
groups and regular groups of automorphisms of l.o. sets and the results obtained were applied to the description
of right-regular representations of right-ordered groups with subnormal systems of convex subgroups.

Glass [114] gave new examples of simple l-groups, which are O-2-transitive groups of automorphisms
without elements with bounded carriers. It is proved [167] that any l-automorphism of the l-group Aut($X$) of
the o-set $X$ is inner. It is established [124] that if Aut($X$) is transitive, then any l-automorphism of Aut($X$) is
induced by an inner automorphism of Aut($X$). If Aut($X$) is O-primitive, then any l-ideal of Aut($X$) is charac-
teristic.

Completely Distributive l-Groups of Automorphisms. In [168] it was established that an l-subgroup $G$
of the l-group Aut($X$) is completely distributive if and only if for any $g, g_0 \in G, a \in I$, the equation $g = \bigcup_{g_0 \in G} g_0$
implies $g = \bigcup_{g \in \text{Aut}(X)} g_0$. Let $G$ be an l-subgroup of Aut($X$), $O(g, x)$ be the supporting interval of the element
$g \in G, x \in X: O(g, x) = \{y \in X | g^n(x) = y \leq g^m(x) \text{ for some integers } n, m\}$. By a compression of $O(g, x)$ is meant
an element $k_g \in \text{Aut}(X)$, such that $k_g(y) = y$ for $y \in O(g, x), k_g(y) = g(y)$ for $y \notin O(g, x)$. $G$ is called a compressed
l-subgroup of Aut($X$) if for any $g \in G, x \in X k_g \in G$. Read [198] noted that any compressed l-subgroup in Aut($X$)
is completely distributive and stabilizers of points in $G$ are closed. In [119] there were studied $\alpha^*$-extensions
of completely distributive subgroups of Aut($X$).

Intransitive l-Groups of Automorphisms. McCleary in [170] carried over the basic concepts and theo-
rems of the theory of transitive l-groups of automorphisms to the case of intransitive l-groups. He introduced
the concept of convex O-block, studied the factorization of intransitive l-groups in relation to the congruence
defined by the partition of $X$ into convex O-blocks. He studied O-primitive intransitive l-subgroups of Aut($X$)
and their connections with transitive O-primitive groups of automorphisms of l.o. sets. The methods devel-
oped are applied to the study of completely distributive l-groups and stabilizers of points.

Groups of automorphisms of l.o. sets were also studied in [123, 16, 49, 50, 51, 52].

4. Topological l-Groups

$tl$-Groups. If on the l-group $G$ there is given a topology, with respect to which the group and lattice
operations are continuous, then $G$ is called an l-group. The topology of a tl-group is called convex if $G$ has
a basis of open sets consisting of convex subgroups of $G$. A closed (in the topological sense) subset of a tl-

group will be called t-closed.

Ball [71] proved that a convex l-subgroup of an O-approximable tl-group is closed if and only if it is t-
closed with respect to any Hausdorff topology on $G$ turning $G$ into a tl-group. An O-approximable l-group has
the weakest Hausdorff topology if and only if it is completely distributive. In [212] there were studied proper-
tries of the lattice of all topologies on an l-group, turning it into a tl-group.

Topological Properties of tl-Groups. One says that the tl-group $G$ has bicompatible origin if in $G$ there
is a neighborhood of the identity with bicompatible t-closure, generating $G$. A. V. Mironov [41-43] proved that
a nilpotent l.o. tl-group of bicompatible origin has a system $G = H_0 \supset H_1 \supset \ldots \supset H_n = K \supset E$ of t-closed convex
subgroups such that $K$ is the connected component of the identity of $G$, which either coincides with $E$, or is

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tl-isomorphic with the additive group $R$ of real numbers with the usual order and topology. G for any $i = 0, 1, \ldots, n$ is a lexicotension of $H_i$ by the l.o. group $G/H_i$ and each $H_i/H_{i+1}$ ($0 \leq i \leq n-1$) is isomorphic with a discrete finitely generated $tl$-subgroup of $R$. If $G$ is not linearly ordered, then $G$ has finite orthogonal rank and is obtained from a finite number of nilpotent l.o. $tl$-groups of bicom pact origin by successive application of the direct product (in the topological and $tl$-group sense) and lexicographic extension. For Abelian $l$-groups the analogous result was obtained by A. N. Islamov [22, 23]. In [24] it is also established that if the Abelian $tl$-group $G$ with convex topology and equal to $\bigcup_{n=1}^{\infty} U_n$ for any neighborhood $U$ of the identity has finite orthogonal rank, then $G$ is $tl$-isomorphic with all of $R^n$. Khudalberdiev [65] considered the connection between topologies and norms on $l$-groups. Pshenichnov [45-47] considered $l$-groups with a topology, continuous at the identity, where the order of $G$ is relatively closed. If $G$ is locally bicom pact, then its connected component is isomorphic with $R^n$. A nilpotent $l$-group of bicom pact origin with relatively closed order is a $tl$-group. The methods obtained are applied to the investigation of Lie groups with lattice order.

5. Ordered Groups Similar to $l$-Groups

Riesz Groups. A partially ordered set $G$ is called a TR($m, n$)-set if it is directed and for any $a_1, \ldots, a_m, b_1, \ldots, b_n \in G$ such that for all $i, j$, $a_i < b_j$, one can find $c \in G$ for which $a_i < c < b_j$ for all $i, j$. By a Riesz group is meant a partially ordered group with the property TR(2, 2). With any Riesz order there is connected a preorder $\subseteq$, compatible with it, defined by the relation: $a \subseteq b$ if and only if for any $a < a$ one has $x < b$ and for any $y > b$ one has $y > a$. A tight Riesz group (TR-group) is a partially ordered group $G$ with the property TR(1, 2), without pseudoidentities, and such that $\subseteq$ is a left directed order of $G$. If $G$ with the order $\subseteq$ is an $l$-group, then $G$ is a TRL-group. A TRL-group $G$ is called androgenous if in it there are elements $x, y$, such that $1) x > e, y < e, x \wedge y = e; 2) the set \{x \in G|x > e, x < e\}$ is nonempty. In [158] there is given a description of TRL-groups, which are $l$-groups with subnormal jumps. Rachunek [193] introduced and studied the concepts of orthogonality and polar in Riesz groups. Now in [195] he described Riesz groups of finite orthogonal rank in terms of l.o. convex subgroups and convex subgroups which are antilattices. Similar results were obtained by Kaminskii [25], who also constructed an example of a Riesz group of finite orthogonal rank, which is not a lexicoproduct of l.o. groups. Ruzicka [205] considered the concept of polar in arbitrary partially ordered groups. Van Meter characterized quasi-$l$-groups $G$ in which each element $g$ can be represented in the form $ab^{-1}$, where $a, b$ are quasiorthogonal. Any quasi-$l$-group is a Riesz group [218].

Reilly [202] defined the concept of hybrid product of partially ordered groups and characterized in terms of subdirect hybrid products, Abelian TRL-groups. Miller [171] extended and improved this description for androgenous TRL-groups.

In [117, 118, 101-103] there are described compatible tight Riesz orders on the $l$-group of automorphisms of a l.o. set. Redfield [201] considered locally compact TRL-groups. He proved that on the $l$-group $G$ there exists a partial order $\preceq$ such that $\langle G, \preceq, \subseteq \rangle$ is a nonandrogenous locally compact TRL-group if and only if $\langle G, \subseteq \rangle$ is the lexicosum of a finite number of additive l.o. groups of real numbers. Miller [172] considered the group of additive functionals, defined on an Abelian TRL-group with the open interval topology with values in $R$, and found conditions under which this group is a TRL-group.

Topological properties of Riesz groups are considered in [66, 209, 210, 220].

Retractions of $l$-Groups. Let $G$ be an $l$-group, $S(G)$ be the semigroup of all subsets of $G$ with respect to the usual multiplication, $E$ be an idempotent in $S(G)$, and $H(E)$ be a maximal subgroup of $S(G)$ containing $E$. In [88] it is proved that if $E$ is a normal idempotent and dual ideal in the lattice $G$, then $H(E)$ is an $l$-group with respect to the order $Q = \{A \in H(E) | A \subseteq E\}$ and $T(E) = \{aE | a \in G\}$ is an $l$-subgroup of $H(E)$, where $aE \vee bE = a \vee b \cdot E$. In [89] it is established that if $G$ is an $O$-approximable $l$-group, then so is $H(E)$, and if $G$ is hyperarchimedean, then $H(E)$ is Archimedean.

For a group $G$, by $F(G)$ we denote the semigroup of finite subsets of $G$ with the usual multiplication and with the operation $\lor: A \lor B = A \cup B$. By a retraction of $G$ is meant a homomorphism of $F(G)$ into $G$. If the set $R(G)$ of all retractions of $G$ is nonempty, then $G$ is called retractable.

In [86] there are given conditions under which a group is retractable, in particular, any $l$-group is such. For $G$ a subgroup $H$ is called a $\sigma$-subgroup if the restriction of $\sigma$ to $F(H)$ is a retraction of $H$. If $G$ is an $l$-group and $\sigma(A) = \operatorname{Sup} A$ for any $A \in F(G)$, then the subgroup $H$ is a $\sigma$-subgroup if and only if $H$ is an $l$-subgroup of $G$. In [85, 86] there is studied the lattice of $\sigma$-subgroups of a retractable group. In [84] there are considered certain concrete examples of retractable groups, in [211] retractions of an l.o. group.
Cyclic and Semihomogeneous Ordered Groups. The set $G$ is called partially cyclically ordered if on $G$ there is given a ternary relation $R$ with the properties: 1) $(a, b, c) \sim R$ implies $a \sim b \sim c \sim a$; 2) $(a, b, c) \equiv R$ implies $(b, c, a) \equiv R$; 3) $(a, b, c) \sim R$ implies $(b, c, a) \sim R$; 4) $(a, b, c) \equiv R$ $(a, c, d) \equiv R$. If for any $a, b, c \in G$, $a \sim b \sim c \sim a$ one of the relations $(a, b, c) \sim R$ or $(a, c, b) \equiv R$ is true, then $G$ is called cyclically ordered. Zheleva [20] studied groups $G$ with cyclic order $R$ such that for any $x, a, b, c \in R$ the relation $(a, b, c) \sim R$ implies $(ax, bx, cx) \sim R$. She calls such groups RCO-groups. Any right-ordered group turns naturally into an RCO-group. Any RCO-group is a quotient group of a suitable right-ordered group by some central cyclic subgroup. Each RCO-group can be represented by automorphisms of a suitable cyclically ordered set.

Bludov and Kokorin [7] considered semihomogeneous lattice ordered groups, i.e., groups on which there is given a lattice order $\leq$ such that for any $g \in G$, $g \neq e$, one of these properties is true: 1) $a < b$ implies $ag < bg$, $ga < gb$ for all $a, b \in G$; 2) $a < b$ implies $ag > bg$, $ga > gb$ for all $a, b \in G$. The group $G$ is semihomogeneously lattice ordered if and only if $G$ has a subgroup $H$ of index 2, which is an $l$-group, while the lattice order of $H$ is normal in $G$. Any semihomogeneous lattice ordered group can be imbedded in a wreath product of an $l$-group and a cyclic group of the second order. A semihomogeneously lattice ordered group can be embeded in the group of automorphisms and antiautomorphisms of an appropriate $l$-o. set.

Further development of this concept was made by the same authors in [8]. By a $T$-genus ordered group is meant a group $G$, on which there are given a partial order relation $\preceq$ and a genus function $\rho : G \to T$, where $T$ is the multiplicative group of complex numbers of modulus 1, such that $\rho(xy) = \rho(x)\rho(y)$; it is assumed that $\rho(x) < \rho(y)$ in $T$, if $\arg x < \arg y$; for any $a, b, x \in G$ the inequality $a < b$ implies: 1) $ax < bx$, $xa < xb$, if $\rho(a) \leq \rho(ax)$ and $\rho(b) < \rho(bx)$ or $\rho(a) > \rho(ax)$ and $\rho(b) < \rho(bx)$. For $T$-genus ordered groups there are proved theorems analogous to assertions on semihomogeneously ordered groups.

$\omega$-Ordered Groups. In the mid-1960s Todorinov introduced the concept of $\omega$-ordered group, i.e., a partially ordered group in which for any element $g \neq e$ which is not generalized periodic, one can find an integer $n$ such that $g^n > e$. The properties of such groups and certain of their generalizations were considered in [58, 60, 61, 63]. For other variants of the definition of orderedness on a group, cf. [17, 44, 174].

6. Extensions of Partial Orders

Vinogradov [15] gave a sufficient condition for the existence on a three-stage solvable group of a non-trivial partial order. Mura and Rhemtulla [178] proved that in any torsion-free group from the manifold any maximal partial order is isolated. Extension of partial orders to interpolational orders was considered by Todorinov [59]. Mura and Rhemtulla [184] constructed examples of orderable nonhyperorderable groups with new interesting properties. Extension of partial orders in systems similar to groups was considered in [192] and [194].

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**MODULES**

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A survey is given of results on modules over rings, covering 1976–1980 and continuing the series of surveys "Modules" in Itogi Nauki.

The present survey covers the materials from the reviewing journal Matematika for 1976–1980 and can be considered as a continuation of the surveys [130, 131, 99, 100, 137] on modules, covering the materials of 1961–1962, 1963–1965, 1966–1968, 1969–1971, and 1972–1975, respectively, and also the survey [97] on rings of endomorphisms. In particular, in citing papers reflected in these surveys, the reader, as a rule, is referred to the corresponding survey paper. References to the bibliography of the surveys cited are formulated as [M62:000], [M65:000], [M68:000], [M71:000], [M75:000], and [KE:000], respectively. For example, [M65:121] means that one has in mind paper 121 from the survey M65 = [131], relating to 1963–1965. References of the type Paragraph 3.1 indicate Section 3 Paragraph 1 of the present survey. In the bibliography as a rule, we do not include reports to conferences or seminars, if the corresponding results have already appeared in the form of journal publications. An exception is made only for survey reports. Definitions of concepts which are in the monographs of Lambek [M71:59] and Faith [161] are usually not given.