PORTRAIT OF THE FOUR REGULAR SUPER-EXPONENTIALS TO BASE SQRT(2)

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ABSTRACT. We introduce the concept of regular super-functions at a fixed point. It is derived from the concept of regular iteration. A super-function F of h is a solution of F(z+1)=h(F(z)). We provide a condition for F being entire, we also give two uniqueness criteria for regular super-functions.

In the particular case h(x)=b^x we call F super-exponential. h has two real fixed points for b between 1 and e^(1/e). Exemplary we choose the base b=sqrt(2) and portrait the four classes of real regular super-exponentials in the complex plane. There are two at fixed point 2 and two at fixed point 4. Each class is given by the translations along the x-axis of a suitable representative.

Both super-exponentials at fixed point 4 — one strictly increasing and one strictly decreasing — are entire. Both super-exponentials at fixed point 2 — one strictly increasing and one strictly decreasing — are holomorphic on a right halfplane. All four super-exponentials are periodic along the imaginary axis. Only the strictly increasing super-exponential at 2 can satisfy F(0)=1 and can hence be called tetrational.

We develop numerical algorithms for the precise evaluation of these functions and their inverses in the complex plane. We graph the two corresponding different half-iterates of h(z)=sqrt(2)^z. An apparent symmetry of the tetrational to base sqrt(2) disproved.

1. SUPER-FUNCTIONS

The functional equation

$$(1.1) \qquad\qquad F(z+1) = h(F(z))$$

refers to the building-up of a super-function F of some given base-function h. Multiplication $F(x) = bx$ is a super-function of addition $h(x) = b + x$. Exponentiation $F(x) = b^x$ is a super-function of multiplication $h(x) = bx$. Such an equation also occurs in the phenomenological description of a nonlinear optical fiber of fixed length [11], where $F(z)$ represents some parameter (for example, the logarithm of the power) at distance z from the input tip, and h is the transfer function. In that application z is supposed to be real. In general, we assume, that equation (1.1) holds for $z \in C \subseteq \mathbb{C}$, and F is holomorphic on C.

Definition 1 (super-function, base-function, super-exponential, tetrational)**.** Let C be a domain such that for each $z \in C$ also $z+1 \in C$. A function F defined on C is called super-function of h, if it satisfies equation (1.1) for all $z \in C$. We call h the *base-function* of F. Applying the prefix $a \mapsto d$ (to "super-function") means that

additionally $F(a) = d$. Sometimes we also include the domain of holomorphism C into the prefix, writing $(C, a \mapsto d)$, which additionally means that F is holomorphic on C. Any super-function of $h(z) = b^z$ is called *super-exponential to base b*. Any $0 \mapsto 1$ super-exponential to base b is called *tetrational to base b*.

The name super-exponential is due to Bromer [6], although other names (generalized exponential, iterated exponential, tetration, ultraexponential, etc.) are also used for the iteration of an exponential function [10, 9, 3, 16].

In order to illustrate the definition of super-function, we suggest few examples below.

Let $h(z) = b{+}z$ on $z \in \mathbb{C}$. Then, the function $F(z) = b{\cdot}z$ on $z \in \mathbb{C}$ is a $(\mathbb{C}, 0 \mapsto 0)$ super-function of h.

Now let $h(z) = b \cdot z$ on $z \in \mathbb{C}$. Then $F(z) = \exp_b(z) = b^z$ is a $(\mathbb{C}, 0 \mapsto 1)$ super-function of h.

Now we let the previous super-function again be the base-function $h = \exp_b$. For $C = \mathbb{C} \backslash \{x \in \mathbb{R} : x \leq -2\}$, the tetrational F by [11] is a $(C, 0 \mapsto 1)$ super-function of exp or super-exponential to base e .

For some base-functions it is possible to obtain elementary super-functions by using special functional relations, for example $F(z) = \cos(\pi 2^z)$ is a $(\mathbb{C}, 0 \mapsto -1)$ super-function of the quadratic base-function $h(z) = 2z^2 - 1$. More trigonometric examples can be found in [18]. Also, for real $j > 0$, and base-function $h(z) = z^j$ the function $F(z) = \exp(j^z)$ is a super-function.

No representation through elementary or special functions is reported for non-constant super-exponentials yet. So it would make sense to lift them into the status of new special functions after clarifying the uniqueness situation (similar to the complex extension of the factorial through the Gamma function).

Without additional conditions a super-function is not unique, generally. If F is a $(C, a \mapsto d)$ super-function of h then for any 1-periodic holomorphic function θ, such that $\theta(a) = 0$, the function $E(z) = F\big(z + \theta(z)\big)$ will be a $(D, a \mapsto d)$ super-function of h for $D = \{z : z + \theta(z) \in C\}$:

$$(1.2) \qquad E(a) \quad = \quad F(a + \theta(a)) = d$$

$$(1.3) \qquad E(z{+}1) \quad = \quad F(z + \theta(z{+}1) + 1) = h(F(z + \theta(z))) = h(E(z)) \ .$$

However, the range D of holomorphism of the function E may be different from C. Here is a question how an appropriately big domain of holomorphism restricts the possible solutions. We give a uniqueness criterion in proposition 7.

This paper is motivated by discussions on [17] about a generalization of the algorithm [11] for a holomorphic tetrational, which seems to work for various values $b > \exp(1/e)$, in particular, for $b = 2$, see [12, 19], but cannot be applied "as is" for the case $1 < b \leq \exp(1/e)$. In order to elaborate a robust algorithm for the evaluation of the tetrational in this case, we consider one specific value $b = \sqrt{2}$ from this interval. We consider not only the tetrational, but also some super-exponentials to this base which can be easily generalized to bases $1 < b < \exp(1/e)$.

In order to be explicit, we look for holomorphic, real solutions F of the equation

$$(1.4) \qquad\qquad F(z{+}1) = \exp_b\big(F(z)\big)$$

for the specific value $b = \sqrt{2}$. (We call any $C \mapsto \mathbb{C}$ function f real, if $f(z^*) = f(z)^* \ \forall z \in C$.)

2. Super-functions by regular iteration

In this section we assume, that the function h has some fixed point λ, id est $h(\lambda) = \lambda$. For example $\lambda = 2$ and $\lambda = 4$ are the only real fixed points of $h = \exp_{\sqrt{2}}$ and $\lambda = e$ is the only real fixed point of $h = \exp_{\exp(1/e)}$. Assume that the function h is holomorphic at least in some vicinity of the fixed point λ. Then, a super-function F of h can be constructed (under certain additional conditions), such that F approaches λ asymptotically, at least in some directions in the complex plane.

The holomorphic function h can be expanded in a series that converges in some domain around the fixed point:

$$(2.1) \qquad f(z) = h(\lambda + z) - \lambda = \sum_{n=1}^{\infty} f_n z^n$$

In this paper, for a function f we use notation f_n for the nth Taylor coefficient at λ. The formal solution F with exponential behavior can be constructed in the following way. Let

$$(2.2) \qquad F(z) = \lambda + G(\mathcal{E}(z)) \quad ,$$

for some function G expandable into a series

$$(2.3) \qquad G(z) = z + \sum_{n=2}^{\infty} G_n z^n$$

and

$$(2.4) \qquad \mathcal{E}(z) = \exp\big((z+t)\,\kappa\big) = s\exp(\kappa z)$$

for some constant parameters t and κ, where $s = e^{\kappa t}$. Substitution $z \mapsto z+1$ in equation (2.2) gives

$$(2.5) \qquad F(z+1) = \lambda + G(K\mathcal{E}(z))$$

where $K = \exp(\kappa)$. Then, equation (1.1) can be written as

$$(2.6) \qquad \lambda + G(K\mathcal{E}(z)) = h\big(\lambda + G(\mathcal{E}(z))\big)$$

$$(2.7) \qquad \lambda + G(K\omega) = h\big(\lambda + G(\omega)\big)$$

The Taylor expansion of both sides of (2.7) at $\omega = 0$ leads to the equalities for the coefficients:

$$(2.8) \qquad K \;=\; f_1$$
$$(2.9) \qquad G_2 K^2 \;=\; f_1 G_2 + f_2$$
$$(2.10) \qquad G_3 K^3 \;=\; f_1 G_3 + 2 f_2 G_2 + f_3$$
$$(2.11) \qquad G_4 K^4 \;=\; f_1 G_4 + 3 f_3 G_2 + 2 f_2 G_3 + f_2 G_2^2 + f_4$$

It seems as if G_n can be determined without solving polynomial equations. This is indeed the case and it turns out that this approach is equivalent to the well-known approach of regular iteration via the Schröder equation [15, 14, 7].

Few notes about notations. For any (formal) powerseries f we denote the coefficients by f_n; $f(z) = \sum_{n=0}^{\infty} f_n z^n$ (although the series has no need to converge.) We

write f^m for the mth power of the function f, and $f^{[m]}$ for the mth iterate of f, id est

$$(2.12) \qquad f^{[m]}(z) = \underbrace{f\Big(f\big(...f(z)...\big)\Big)}_{m \text{ evaluations of } f}$$

For example, below $f^m{}_n$ means the nth Taylor coefficient of the function that is the mth power of function f. It should not be confused with $f_n{}^m$, which is the mth power of the nth Taylor coefficient, or with $f^{[m]}{}_n$, which is the nth coefficient of the expansion of the mth iterate of function f at 0. There are few exceptions from this rule. First, we keep the usual notation F^{-1} for the inverse function of F, although it is actually minus first iteration of function F; and of course we use subscripts as usual on non-functions, like κ_2 or T_4. You can find a table of notations in the appendix; we tried to distribute the semantic load uniformly among the letters of Latin alphabet, but use few Greek letters too.

Definition 2 (Schröder equation, Schröder function)**.** For a function h with fixed point λ and $K = h'(\lambda)$, we call any function χ that satisfies the Schröder equation

$$(2.13) \qquad \chi(h(z)) = K\chi(z)$$

a *Schröder function* of h at λ.

For ease of derivation we first assume that the fixed point is 0. If we want the Schröder function χ to be analytic, then it is already determined by $\chi'(0) = 1$. We show this in the next proposition.

Proposition 1 (Schröder powerseries)**.** *Let f be a formal powerseries with $f_0 = 0$ (fixed point 0) and $f_1{}^n \neq f_1$ for all integer $n > 1$ (particularly if $|f_1| \neq 0, 1$). Then there is a unique formal powerseries solution χ with $\chi_1 = 1$ of the Schröder equation*

$$(2.14) \qquad \chi(f(z)) = f_1\chi(z).$$

We call this solution the Schröder powerseries *of f. The Schröder powerseries χ and its inverse $\eta = \chi^{-1}$ are given by $\chi_0 = \eta_0 = 0$, $\chi_1 = \eta_1 = 1$, and, for $n > 1$, by the recursive formulas*

$$(2.15) \qquad \chi_n = \frac{1}{f_1 - f_1{}^n} \sum_{m=1}^{n-1} \chi_m f^m{}_n, \qquad \eta_n = \frac{1}{f_1{}^n - f_1} \sum_{m=2}^{n} f_m \eta^m{}_n$$

where the mth power of an arbitrary formal powerseries g is defined by

$$(2.16) \qquad g^m{}_n = \sum_{n_1 + \cdots + n_m = n} g_{n_1} \cdots g_{n_m}.$$

Proof. The recursive solutions are a direct consequence of the composition formula of two formal powerseries f and g with $g_0 = 0$:

$$(2.17) \qquad (f \circ g)_n = \sum_{m=0}^{n} f_m g^m{}_n$$

applied to our Schröder equation (2.14) or its inverse $\eta(f_1 z) = f(\eta(z))$:

$$f_1 \chi_n = \sum_{m=0}^{n} \chi_m f^m{}_n \quad , \qquad\qquad f_1{}^n \eta_n = \sum_{m=1}^{n} f_m \eta^m{}_n.$$

First we can observe at the left side that $f_1\chi_0 = \chi_0$ which implies $\chi_0 = 0$ for $f_1 \neq 0, 1$. In consequence $\eta_0 = 0$. For the indices above 1 we rearrange the equations to get the χ_n or η_n that occur in the sum of the right side to the left side and where we take into account that $f^n{}_n = f_1{}^n$ and $\eta^1{}_n = \eta_n$. $\qquad \square$

Proposition 2. *Every formal powerseries $\mathcal{X}$ solution of (2.14) can be expressed as $\mathcal{X}(z) = s\chi(z)$ for some $s \in \mathbb{C}$.*

Proof. Setting $\mathcal{X}_1 = s$ in the above derivation, one inductively gets $\mathcal{X}_n = s\chi_n$. $\quad \square$

Now we apply this to our original problem (2.7). We see that G is just the inverse Schröder function of $h(z+\lambda) - \lambda$. Let us write down the first few coefficients of G.

$$G_0 = 0 \quad , \qquad\qquad\qquad G_1 = 1$$

(2.18)
$$G_2 = \frac{1}{f_1^2 - f_1} f_2 g^2{}_2 = \frac{f_2}{f_1^2 - f_1}$$

$$G_3 = \frac{f_2 \left(\sum_{n_1+n_2=3} G_{n_1} G_{n_2}\right) + f_3 G_3{}^3}{f_1^3 - f_1} = \frac{f_2 (2G_1 G_2) + f_3}{f_1^3 - f_1}$$

(2.19)
$$= \frac{2 f_2 \frac{f_2}{f_1^2 - f_1} + f_3}{f_1^3 - f_1} = \frac{2 f_2 f_2 + f_3 f_1^2 - f_3 f_1}{(f_1^3 - f_1)(f_1^2 - f_1)} = \frac{2 f_2^2 + f_3 f_1^2 - f_3 f_1}{(f_1-1)(f_1^2-1) f_1^2}$$

Proposition 3. *If the powerseries f with $f_0 = 0$ and $|f_1| \neq 0, 1$ has non-zero convergence radius, then the Schröder powerseries χ of f has non-zero convergence radius too.*

Proof. For a proof you can see [14], theorem 4.6.1. $\qquad\qquad \square$

Proposition 4. *Let f be holomorphic in a vicinity of 0, $f(0) = 0$, set $K = f'(0)$, let $|K| \neq 0, 1$. If f is entire and $|K| > 1$ or if f^{-1} is entire and $|K| < 1$ then the inverse Schröder function of f (i.e. the function with the inverse Schröder powerseries development at 0) is entire.*

Proof. We continue the inverse Schröder function to the whole complex plane. By proposition 3 there is a disk D_δ with radius $\delta > 0$ at 0 where the inverse Schröder function η of f is analytic. We then define $\hat{\eta}$ on the disk with radius $|K|^{\pm 1} \delta$ for all $z \in D_\delta$ via

(2.20) $\qquad\qquad \hat{\eta}(Kz) := f(\eta(z)) \qquad\qquad$ for $\quad |K| > 1$

(2.21) $\qquad\qquad \hat{\eta}(z) := f^{-1}(\eta(Kz)) \qquad\qquad$ for $\quad |K| < 1$

$\hat{\eta}$ coincides by the Schröder equation with η on D_δ and is hence a continuation to $D_{|K|^{\pm 1}\delta} \supset D_\delta$. We repeatedly apply this process to continue η to the disk with radius $|K|^{\pm n} \delta$ and so continue η to the whole complex plane. $\qquad \square$

Proposition 5. *Let f be analytic at the fixed point 0, set $K = f'(0)$ and assume $|K| \neq 0, 1$. Then, the Schröder powerseries χ of f can be analytically continued to the basin of attraction $\Omega(f) = \{z \colon \lim_{n\to\infty} f^{[n]}(z) = 0\}$ of f at 0 (for $|K| < 1$) or to $\Omega(f^{-1})$ (for $|K| > 1$), and the convergence radius R of χ is bounded by*

$$R \geq \sup \left\{r > 0 \colon B_r(0) \subseteq \Omega(f^{[\pm 1]})\right\}$$

Proof. By the equation $\chi\left(f^{[n]}(z)\right) = K^n\chi(z)$ we get a rule to continue χ to $\Omega(f^{\pm 1})$:

$$\chi(z) = \frac{\chi\left(f^{[\pm n]}(z)\right)}{K^{\pm n}}.$$

$\square$

We saw that we can express a super-function F of f through the inverse Schröder function: $F(z) = \eta(se^{kz})$. As it is mentioned in the introduction, there is wide range of other $a \mapsto d$ super-functions: If F is a holomorphic $a \mapsto d$ super-function of f then for every holomorphic 1-periodic function θ with $\theta(c) = 0$ the function $E(z) = F(z+\theta(z))$ is another holomorphic $a \mapsto d$ super-function of f. Therefore, a condition, that singles out the solutions $\eta(se^{\kappa z})$, would be quite welcome. Such a condition is given in the next two propositions.

Proposition 6. *Let κ be a complex number with $\Re(\kappa) \neq 0$. Then for each $s \in \mathbb{C}$ the function $F(z) = se^{\kappa z}$ is the only $0 \mapsto s$ solution of $F(z+1) = e^\kappa F(z)$, that is holomorphic and bounded on some strip $S_{x,\varepsilon} = \{z/\kappa + x - \xi\colon 0 \leq \xi < 1+\varepsilon,\ \Re(z) = 0\}$ (which is the strip between the straight lines orthogonal to $\overline{\kappa}$ going through the points x and $x-1-\varepsilon$).*

Proof. First $F(z) = se^{\kappa z}$ is indeed a $0 \mapsto s$ solution and it is bounded on $w \in S_{x,\varepsilon}$ for any fixed x and $\varepsilon > 0$ because $|se^{\kappa w}| = |s|\,e^{\Re(z+\kappa(x-\xi))} = |s|\,e^{\Re(\kappa)(x-\xi)} \leq |s|\,e^{|\Re(\kappa)|(1+\varepsilon+|x|)}$.

$F_1(z) := e^{\kappa z}$ is also bounded from below by $|F_1(z)| \geq e^{-|\Re(\kappa)|(|x|+1+\varepsilon)}$. Let $\tilde{F}$ be another solution, consider $\theta(z) = \tilde{F}(z)/F_1(z)$ which must be bounded on $S_{x,\varepsilon}$. Then $\theta(z+1) = \theta(z)$, θ is a periodic function on $S_{x,\varepsilon}$. The translations of the strip $S_{x,\varepsilon}$ by integers k overlappingly cover $\mathbb{C}$, $\cup_{k\in\mathbb{Z}}(S_{x,\varepsilon}+k) = \mathbb{C}$. (Note that this is not the case for $\Re(\kappa) = 0$.) Then, function θ can be holomorphically continued to $\mathbb{C}$ by $\theta(z+k) = \theta(z)$. Now θ is entire and bounded and hence constant: $\tilde{F}(z) = sF_1(z)$, where s must be $\tilde{F}(0)$. $\square$

The following proposition characterizes the solutions $\eta(se^{\kappa z})$ as those which satisfy a certain asymptotic property in the direction κ of the complex plane.

Proposition 7. *Let f be analytic at 0, $f(0) = 0$, $K := f'(0)$, and $|K| \neq 0, 1$, let κ be some number with $e^\kappa = K$; and let $H_{\kappa,x} := \{z/\kappa\colon \Re(z) \leq x\}$ be the halfplane bounded by the straight line going through and being orthogonal to x/κ. Then the set of on any $H_{\kappa,x}$ holomorphic and bounded super-functions F of f which satisfy $\lim_{x\to\infty}\sup|F(H_{\kappa,-x})| = 0$ consists exactly of the functions*

(2.22)
$$F(z) = \eta(se^{\kappa z}), \quad s \in \mathbb{C},$$

where η is the inverse Schröder powerseries of f.

Proof. First, each $F(z) = \eta(se^{\kappa z})$ is clearly a holomorphic super-functions of f. It is bounded, because $se^{\kappa z}$ maps $H_{\kappa,x}$ to the closed disk $|z| \leq |s|\,e^x$, where η is holomorphic (continuous) and hence bounded.

Now let F be an arbitrary on $H_{\kappa,x}$ holomorphic and bounded super-function of f which satisfies $\lim_{x\to\infty}\sup|F(H_{\kappa,-x})| = 0$; consider the function $g(z) = \eta^{-1}(F(z))$. Choosing x small enough, by $\lim_{x\to\infty}\sup|F(H_{\kappa,-x})| = 0$ we can ascertain that $F(z)$ has values only within the convergence disk of η^{-1} for $z \in H_{\kappa,x}$. So g is holomorphic, and bounded on $H_{\kappa,x}$, and satisfies

$$g(z+1) = \eta^{-1}(f(F(z))) = K\eta^{-1}(F(z)) = Kg(z)$$

for all z such that $z, z+1 \in H_{\kappa,x}$. As $|K| \neq 1$, we know that $\Re(\kappa) \neq 0$; and from proposition 6 it follows, that $g(z) = se^{\kappa z}$ and hence $F(z) = \eta(se^{\kappa z})$. $\qquad\square$

Now we switch from the fixed point 0 to an arbitrary fixed point λ. If λ is a fixed point of h, then 0 is a fixed point of $z \mapsto h(z+\lambda) - \lambda$. It is clear, that, if χ is a Schröder function (meaning that is satisfies the Schröder equation) of h, then $z \mapsto \chi(z+t)$ is a Schröder function of $z \mapsto h(z+t) - t$ for any t;

$$\chi\Big(\big(h(z+t) - t\big) + t\Big) = \chi\big(h(z+t)\big) = K\chi(z+t)$$

and vice versa: if η is an inverse Schröder function of h, then $z \mapsto \eta(z) - t$ is an inverse Schröder function of $z \mapsto h(z+t) - t$:

$$\eta(Kz) - t = h\big(\eta(z)\big) - t = h((\eta(z) - t) + t) - t .$$

Definition 3 (regular and principal super-function)**.** Let h be analytic at its fixed point λ and $e^{\kappa} = h'(\lambda)$, $\Re(\kappa) \neq 0$. We call the functions $z \mapsto \eta(se^{\kappa z}) + \lambda$, $s \in \mathbb{C}$, the κ-*regular super-functions of h at* λ, where η is the inverse Schröder powerseries of $z \mapsto h(z+\lambda) - \lambda$. We call the function $z \mapsto \eta(e^{\kappa z}) + \lambda$ the κ-*principal* super-function of h at λ. Omitting κ above means $\kappa = \log(h'(\lambda))$ for the standard branch of the logarithm.

Corollary 8. *Every κ-regular super-function F is periodic with period $T = 2\pi i/\kappa$ and $F(z)$ is a translation of the κ-principal super-function $\mathcal{F}(z)$ along the z-plane, i.e. $F(z) = \mathcal{F}(z+t)$ for some $t \in \mathbb{C}$. We can always choose $t = \log(s)/\kappa$, where s is taken from the previous definition.*

Corollary 9. *Let h be analytic at λ, $h(\lambda) = \lambda$, $K := h'(\lambda)$, such that $|K| \neq 0, 1$, let κ be some number with $e^{\kappa} = K$; and let $H_{\kappa,x}$ as in proposition 7. Then the set on some $H_{\kappa,x}$ holomorphic and bounded super-functions F of h which satisfy $\lim_{x\to\infty} \sup |F(H_{\kappa,-x}) - \lambda| = 0$ is the set of the κ-regular super-functions of h at λ.*

Let us now consider strictly increasing real-analytic super-functions. We know that the inverse Schröder function of a real-analytic function h is strictly increasing in a vicinity of the fixed point because $\eta'(0) = 1$, and we want to define the super-function as $F(x) = \eta(se^{\kappa x}) + \lambda$.

There is a natural distinction between positive and negative s. (We exclude the trivial case $s = 0$, which gives the constant function $F(x) = \lambda$). Different s with the same sign just translate the graph of F along the x-axis; $\pm|s|\, e^{\kappa x} = \pm e^{\kappa(x+\ln|s|/\kappa)}$. The change of sign of s is more dramatic; it corresponds to a translation along the imaginary axis and inverses the monotony. We use this for a classification of the regular super-functions of real base-functions:

Definition 4 (super-function above and below)**.** Let h be real-analytic at the fixed point $\lambda = h(\lambda)$ such that $0 < h'(\lambda) \neq 1$. Let $\kappa = \ln(h'(\lambda))$. Let η be the inverse Schröder powerseries of $x \mapsto h(x+\lambda) - \lambda$. Then, we call the function

$$F(z) = \eta(se^{\kappa z}) + \lambda$$

super-function of h *below* λ, for each $s < 0$; super-function *equal to* λ (i.e. the constant function $F(z) = \lambda$), for $s = 0$; and super-function *above* λ, for each $s > 0$.

Let I be the maximal interval around λ such that η is analytically continuable to I and strictly increasing there. For any $d \in \eta(I) + \lambda$ and

$$(2.23) \qquad\qquad s = \eta^{-1}(d - \lambda)$$

we call the function

$$(2.24) \qquad\qquad F(z) = \eta(se^{\kappa z}) + \lambda$$

the regular $0 \mapsto d$ super-function F of h at λ.

The reader may verify that these functions are indeed all possible real-analytic regular super-functions.

Let $F(x) = \eta(se^{\kappa x}) + \lambda$, i.e. as in the previous definition. Let the word "finally" mean: "there exist an x_0 such that for all $x > x_0$" in the case $\kappa < 0$, and "there exist an x_0 such that for all $x < x_0$" in the case $\kappa > 0$. Then we can apply our previous propositions and state:

(1) For all $\varepsilon > 0$ finally $|F(x) - \lambda| < \varepsilon$.
(2) Finally $F(x) > \lambda$ if $s > 0$.
(3) Finally $F(x) < \lambda$ if $s < 0$.
(4) Finally F is strictly increasing if $\kappa s > 0$.
(5) Finally F is strictly decreasing if $\kappa s < 0$.
(6) F is holomorphic on a right halfplane if $\kappa < 0$.
(7) F is entire if $\kappa > 0$.
(8) F is periodic with the imaginary period $T = 2\pi i/\kappa$.

Actually the conditions 1, 6 and 8 already determine F to be a regular super-function, as we show in the next proposition for the case $\kappa < 0$. This is a supplemental uniqueness criterion to the one given in proposition 7 and corollary 9.

Proposition 10. *Let h be a function, real-analytic in a vicinity of its fixed point λ, with $0 < h'(\lambda) < 1$; let $\kappa = \ln(h'(\lambda))$. Each real-analytic super-function F holomorphic on a right half-plane $H_x = \{z \in \mathbb{C} \colon \Re(z) > x\}$ and $2\pi i/\kappa$ periodic that satisfies $\lim_{x \to \infty} F(x) = \lambda$ is a regular super-function.*

Proof. Let $G(z) = \eta(e^{\kappa z}) + \lambda$ be the principal super-function of h. It is finally either strictly increasing or strictly decreasing, hence finally injective. As $\lim_{x \to \infty} F(x) = \lambda$ there is an x_0 such that

$$(2.25) \qquad\qquad W(x) = G^{-1}(F(x))$$

is well-defined for $x > x_0$. It satisfies

$$(2.26) \qquad\qquad W(x+1) = G^{-1}(h(F(x)) = W(x) + 1$$

and we can express F with G and W for $x > x_0$:

$$(2.27) \qquad\qquad F(x) = G(W(x)) = \eta(e^{\kappa W(x)}) + \lambda$$

Now we have the precondition that F is $2\pi i/\kappa$ periodic. We know also that $\eta(x)$ is injective in a vicinity ε of 0. Hence we possibly increase x_0 somewhat so that $e^{\kappa x} < \varepsilon$ for $x > x_0$. We apply then the injectivity of η on the periodic equality:

$$\eta\left(\exp\left(\kappa W\left(x + 2\pi i/\kappa\right)\right)\right) + \lambda = \eta\left(\exp\left(\kappa W\left(x\right)\right)\right) + \lambda$$

$$\exp(\kappa W\left(x + 2\pi i/\kappa\right)) = \exp(\kappa W\left(x\right))$$

$$\kappa W(x + 2\pi i/\kappa) = \kappa W(x) + 2\pi i N$$

for some integer N. So W has the property

$$(2.28) \qquad W\left(x + \frac{2\pi i}{\kappa}\right) = W(x) + \frac{2\pi i}{\kappa} N$$

By (2.26) $\theta(x) = W(x) - x$ is 1-periodic. It has a Fourier-development, which means that there is a Laurent series ρ at 0 such that

$$W(z) - z = \theta(z) = \rho\left(e^{2\pi i z}\right)$$
$$W(z) = \rho\left(e^{2\pi i z}\right) + z.$$

Now we plug this into equation (2.28) and derive

$$\rho\left(e^{2\pi i (z + 2\pi i/\kappa)}\right) + z + \frac{2\pi i}{\kappa} = \rho\left(e^{2\pi i z}\right) + z + \frac{2\pi i}{\kappa} N$$
$$\rho\left(e^{2\pi i z} e^{-4\pi^2/\kappa}\right) = \rho\left(e^{2\pi i z}\right) + \frac{2\pi i}{\kappa}(N - 1)$$
$$\rho\left(z e^{-4\pi^2/\kappa}\right) = \rho(z) + \frac{2\pi i}{\kappa}(N - 1)$$

As the Laurent series development is unique we obtain the equations

$$\rho_n e^{-4n\pi^2/\kappa} = \rho_n$$

for $n \neq 0$ which implies $\rho_n = 0$ and

$$\rho_0 = \rho_0 + \frac{2\pi i}{\kappa}(N - 1)$$

which implies $N = 1$. Hence $\rho(x) = \rho_0$ is constant, then $W(x) = x + \rho_0$, then $F(x) = G(x + \rho_0)$, which actually means that F is a regular super-function of h, see definition 3. $\qquad \square$

In the sequel we apply the propositions and statements before to $h = \exp_{\sqrt{2}}$ with the two fixed points $\lambda = 2, 4$.

3. Super-exponential below fixed point 2

For base $b > \exp(1/e)$ a super-exponential can be evaluated through the Cauchy integral, that gives the integral equation for values of the function along the imaginary axis [11]. For $b = \sqrt{2}$, this algorithm cannot be applied as is, because both fixed points $\lambda = 2$ and $\lambda = 4$ are real; but regular iteration can be applied for precise evaluation.

Throughout this section, the base is $b = \sqrt{2}$, $h = \exp_b$; we consider the fixed point $\lambda = 2$, so,

$$(3.1) \qquad K_2 = h'(\lambda) = \exp_{\sqrt{2}}{}'(2) = 2\ln\left(\sqrt{2}\right) = \ln(2) \approx 0.69314718055995$$

$$(3.2) \qquad \kappa_2 = \ln(K_2) = \ln\left(\ln(2)\right) \approx -0.36651292058166 \ .$$

With formula (2.15) we compute the coefficients of the inverse Schröder function

$$(3.3) \qquad v(z) = z + \sum_{n=2}^{\infty} v_n z^n$$

TABLE 1. Evaluation of coefficients v and V in (3.3), (3.18)

n	v_n	V_n
1	1.0000000000000000	1.0000000000000000
2	−0.5647228383177324	0.5647228383177324
3	0.3381775868511833	0.2996461813840881
4	−0.2103313021386278	0.1559323904892543
5	0.1344548790521098	0.0803518797481544
6	−0.0877843886012191	0.0411584960662439
7	0.0582880930830947	0.0209985209544120
8	−0.0392407117837278	0.0106825803202636
9	0.0267232860342981	0.0054228810223159
10	−0.0183765205976376	0.0027482526618683
11	0.0127420898467766	0.0013909151872678
12	−0.0088986329515697	0.0007031815862125
13	0.0062531995639749	0.0003551700677648
14	−0.0044181328624397	0.0001792537427482
15	0.0031365295362696	0.0000904088765718
16	−0.0022361213774487	0.0000455725430285
17	0.0016001999145218	0.0000229602263218
18	−0.0011489818761273	0.0000115627707503
19	0.0008274921384317	0.0000058201696570
20	−0.0005975832172069	0.0000029289688393

of

$$h[2](z) \quad := \quad \exp_{\sqrt{2}}(z+2) - 2 = 2\left(\exp\left(\frac{\ln(2)}{2}z\right) - 1\right)$$

$$(3.4) \qquad h[2](z) \quad = \quad \sum_{n=1}^{\infty} \frac{\ln(2)^n}{2^{n-1}n!} z^n,$$

$$(3.5) \qquad h[2]_1 \quad = \quad \ln(2),$$

$$(3.6) \qquad h[2]_2 \quad = \quad \frac{\ln(2)^2}{4},$$

$$(3.7) \qquad h[2]_3 \quad = \quad \frac{\ln(2)^3}{24},$$

and so on. As example we compute v_2 and v_3 via (2.18) and (2.19):

$$(3.8) \qquad \begin{aligned} v_2 &= \frac{\ln(2)/4}{\ln(2)-1} & \approx -0.56472283831773236365 \\ v_3 &= \frac{\ln(2)^2(2+\ln(2))/24}{(\ln(2)-1)(\ln(2)^2-1)} & \approx 0.33817758685118329988 \end{aligned}$$

More values for the coefficients v are shown in Table 1.

The regular $0 \mapsto 1$ super-exponential is then by definition 4

$$(3.9) \qquad F_{2,1}(z) = v\left(s_{2,1}e^{\kappa_2 z}\right) + \lambda = v\left(s_{2,1}\ln(2)^z\right) + 2$$

$$(3.10) \qquad s_{2,1} = V(1-\lambda) = V(-1) \approx -0.63209866105083$$

where V is the inverse powerseries of v (i.e. the Schröder powerseries of $h[2]$). Because κ_2 is negative the function $F_{2,1}(z)$ is holomorphic on some right halfplane (see the statement list at the end of section 2).
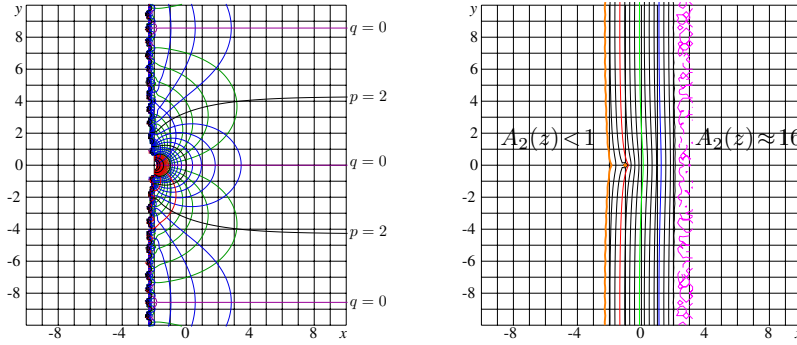
FIGURE 1. Function $f = \tilde{F}_{2,1,20}(z)$ by (3.11) in the $z = x + iy$ plane: levels $p = \Re(f) = $const and levels $q = \Im(f) = $const are shown with thick lines for integer values of p and q; left. Agreement $f = A_2(z)$ by (3.12) in the $z = x + iy$ plane; levels $f = 1, 4, 8, 12, 16$ are shown with thick lines; right.

The truncation of the series (3.3), keeping the last term at index N gives a function, let us call it $\tilde{F}_{2,1,N}$, which approximates $F_{2,1}(z)$ at large positive values of $\Re(z)$:

$$(3.11) \qquad \tilde{F}_{2,1,N}(z) = 2 + \sum_{n=1}^{N} v_n s_{2,1}^n \ln(2)^{zn}$$

For $N = 20$, this function is shown in the left hand side picture of figure 1 with lines of constant real part and those of constant imaginary part. In the figure, the level $p = 1$ visually goes through the origin of coordinates; in reality $\tilde{F}_{2,1,20}(0)$ slightly deviates from unity.

The precision of the approximation can be characterized with the agreement

$$(3.12) \qquad A_2(z) = -\lg \left| \frac{\tilde{F}_{2,1,N}(z) - \log_b\big(\tilde{F}_{2,1,N}(z+1)\big)}{\tilde{F}_{2,1,N}(z) + \log_b\big(\tilde{F}_{2,1,N}(z+1)\big)} \right|,$$

Roughly, this agreement indicates, how many significant figures we may expect to get with this approximation. In the left hand side of the picture, the agreement is smaller than unity; even the first decimal digit of the approximation is doubtful. In the right hand side, the agreement is of order of 16. We expect, at $\Re(z) > 3$ the precision of approximation $\tilde{F}_{2,1,20}(z)$ is limited mainly by the rounding errors. Even in vicinity of the origin of coordinates, the approximation $\tilde{F}_{2,1,20}$ gives several correct decimal digits.

To evaluate $F_{2,1}(z)$ for various values z, the integer iterations can be used:

$$(3.13) \qquad F(z-k) = \log_b^{[k]}(F(z))$$

For the complex$\langle$double$\rangle$ implementation of $F_{2,1}$, we suggest the approximation

$$(3.14) \qquad \tilde{F}_{2,1}(z) = \begin{cases} \tilde{F}_{2,1,20}(z) & , \quad \Re(z) > 3 \\ \log_b\big(\tilde{F}_{2,1}(z+1)\big) & , \quad \Re(z) \le 3 \end{cases}$$

This approximation is estimated to return at least 14 significant decimal digits; it is used to plot function $F_{2,1}$ in figure 2 in the same notations, as in figure 1.   In the
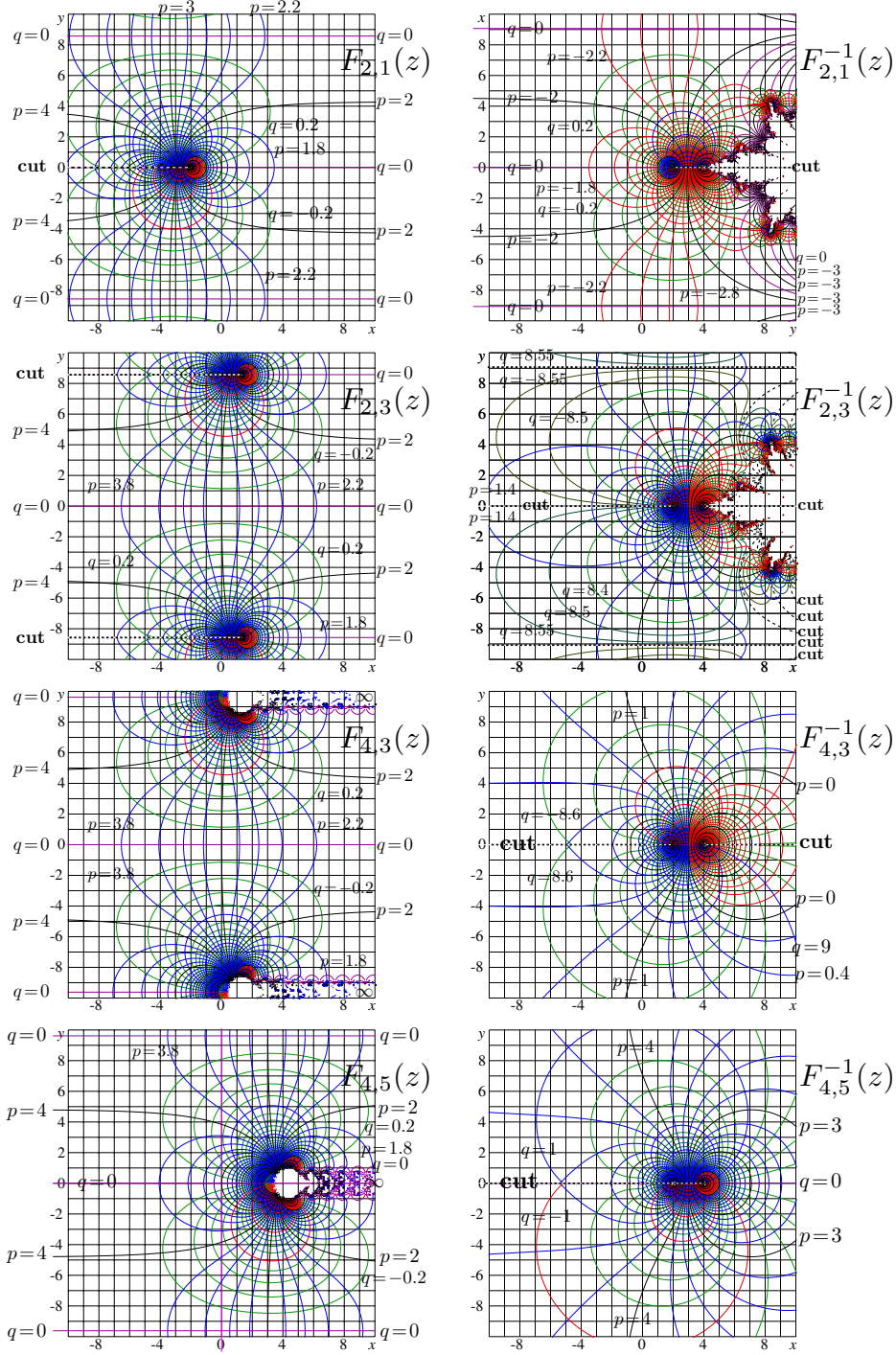
FIGURE 2. Functions $f = F_{2,1}(z)$, $f = F_{2,3}(z)$, $f = F_{4,3}(z)$, $f = F_{4,4}(z)$, left, and their inverse functions, right, in the $z = x + \mathrm{i}y$ plane, in the same notations as in fig.1.
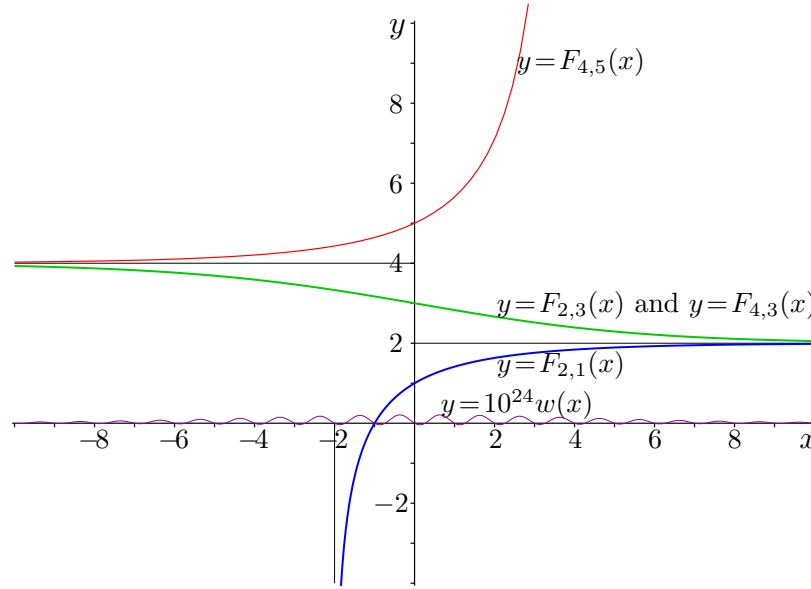
FIGURE 3. $F_{2,1}$ by (3.9), $F_{2,3}$ by (5.1) (which coincides with $F_{4,3}$ by (5.2) with 24 significant figures), and $F_{4,5}$ by (4.9). The asymptotes are shown with thin lines. The difference w by (5.11) scaled with factor 10^{24} is shown with the oscillating curve in vicinity of the abscissa axis.

right hand side of the plot, the function coincides with the primary approximation $\tilde{F}_{2,1,20}$. The approximation also reproduces the periodicity of $F_{2,1}$:

$$(3.15) \quad F_{2,1}(z+T_2 n) = F_{2,1}(z) , \quad T_2 = \frac{2\pi \mathrm{i}}{\kappa_2} \approx -17.1431481793548471 i , \quad n \in \mathbb{Z} .$$

The function $F_{2,1}(z)$ has a branchpoint with logarithmic singularity at $z = -2$; we place the cut along the line $z < -2$. Due to the periodicity, there is a set of singularities and cut lines translated with period T_2 by (3.15). The other pictures in the figure 2 represent functions considered in the following sections.

In figure 3 we plot the function $F_{2,1}$ versus real argument; see the thick curve at the bottom. The other curves correspond to the super-exponentials which we define and analyze in the following sections. We indicate the fixed point λ as first subscript of function F, and its value at zero as the second subscript. We choose this value as the natural number, closest to the fixed point. In such a way, we deal with the following super-exponentials:

$F_{2,1}$, developed at $\lambda = 2$, with range of values $(-\infty, 2)$ along the real axis,
$F_{2,3}$, developed at $\lambda = 2$, with range of values $(2 , 4)$ along the real axis,
$F_{4,3}$, developed at $\lambda = 4$, with range of values $(2 , 4)$ along the real axis,
$F_{4,5}$, developed at $\lambda = 4$, with range of values $(4 , \infty)$ along the real axis,
and their inverse functions. All these functions are plotted in figure 2; their behavior along the real axis is shown in figure 3.

3.1. **The inverse super-exponential below 2.** The inverse function of $F_{2,1}(z) = v(s_{2,1}e^{\kappa_2 z}) + \lambda$ is given by

$$(3.16) \qquad F_{2,1}^{-1}(z) = \log\left(V(z-\lambda)/s_{2,1}\right)/\kappa_2$$

where V is the Schröder powerseries of $h[2](z) := h(z+2) - 2 = 2(\sqrt{2}^z - 1)$. The function $F_{2,1}^{-1}$ is shown in the top of the right hand side of the figure 2. We calculate the first few coefficients according to formula (3.7) and (2.15):

$$(3.17) \quad \begin{array}{rcl}
V_1 = & 1 & \\
V_2 = & -v_2 = \frac{1}{4}\frac{\ln(2)}{1-\ln(2)} & \approx 0.56472283831773236365 \\
V_3 = & \frac{\ln(2)^2}{24}\frac{1+2\ln(2)}{(1-\ln(2))^2(1+\ln(2))} & \approx 0.29964618138408807683
\end{array}$$

More values are printed in the right column of the table 1.

The series can be truncated, keeping the term of the Nth power, giving the approximation

$$(3.18) \qquad \tilde{F}_{2,1,N}^{-1}(z) = \frac{1}{\kappa_2}\log\left(\frac{1}{s_{2,1}}\sum_{n=1}^{N} V_n \cdot (z-2)^n\right).$$

It is useful at small values $|z-2|$. The range of approximation can be extended with the function

$$(3.19) \qquad \tilde{F}_{2,1}^{-1}(z) = \begin{cases} \tilde{F}_{2,1,N}^{-1}(z) & , \quad |z-2| \leq \rho \\ \tilde{F}_{2,1}^{-1}\left(\exp_b(z)\right) - 1 & , \quad |z-2| > \rho \end{cases}$$

In particular, for $N=20$ and $\rho=0.4$, the approximation $\tilde{F}_{2,1}^{-1}$ returns of order of 14 correct decimal digits and can be used for the complex$\langle$double$\rangle$ implementation in numerical analysis.

The plots of functions $F_{2,1}$ and $F_{2,1}^{-1}$ look similar. This similarity indicates approximate symmetry $x \leftrightarrow -y$ of the graph $y = F_{2,1}(x)$; this function is shown again in left hand picture of the figure 7; indeed, it looks pretty symmetric. We analyze this visual effect in section 8.

4. Super-exponential above fixed point 4

In this section is

$$b = \sqrt{2}, \quad h(z) = \exp_b(z), \quad \lambda = 4$$

$$(4.1) \qquad K_4 = h'(\lambda) = \exp_{\sqrt{2}}'(4) = 4\ln\left(\sqrt{2}\right) = 2\ln(2) \approx 1.3862943611199$$

$$(4.2) \qquad \kappa_4 = \ln(K_4) \approx 0.32663425997828098238.$$

With formula (2.15) we compute the powerseries u of the inverse Schröder function of $h[4](z) := h(z+4) - 4$

$$(4.3) \qquad h[4](z) = 4\left(\exp\left(\frac{\ln(2)}{2}z\right) - 1\right) = 2h[2](z)$$

$$(4.4) \qquad h[4](z) = \sum_{n=1}^{\infty} \frac{\ln(2)^n}{2^{n-2}n!}z^n$$

$$(4.5) \qquad h[4]_1 = 2\ln(2), \; h[4]_2 = \frac{\ln(2)^2}{2}, \; h[4]_3 = \frac{\ln(2)^3}{12}, \ldots$$

TABLE 2. Coefficients u and U in expansions (4.9) and (4.22)

n	u_n	U_n
1	1.0000000000000000	1.0000000000000000
2	0.4485874311952612	-0.4485874311952612
3	0.1903722467978068	0.2120891200549197
4	0.0778295765369683	-0.1021843675069717
5	0.0309358603057080	0.0496986830373718
6	0.0120221257690659	-0.0243075903261196
7	0.0045849888965617	0.0119330883965109
8	0.0017207423310577	-0.0058736976420089
9	0.0006368109038799	0.0028968672871058
10	0.0002327696003030	-0.0014309081060793
11	0.0000841455118381	0.0007076637148566
12	0.0000301156464937	-0.0003503296158730
13	0.0000106807458130	0.0001735756004664
14	0.0000037565713616	-0.0000860610119291
15	0.0000013111367785	0.0000426959089013
16	0.0000004543791625	-0.0000211930290682
17	0.0000001564298463	0.0000105244225996
18	0.0000000535232764	-0.0000052285174354
19	0.0000000182077863	0.0000025984499916
20	0.0000000061604765	-0.0000012917821121

(4.6) $\qquad u_2 = \qquad \frac{\ln(2)/4}{2\ln(2)-1} \qquad \approx 0.44858743119526122890$

(4.7) $\qquad u_3 = \quad \frac{\left(1+\ln(2)\right)\ln(2)^2/12}{1-2\ln(2)-4\ln(2)^2+8\ln(2)^3} \quad \approx 0.19037224679780675668$

More values for the coefficients u are printed in table 2.

While in the case $\lambda = 2$ the preferable initial condition is $F_{2,1}(0) = 1$, which implies

$$(4.8) \qquad F_{2,1}(n) = \exp_{\sqrt{2}}^{[n]}(1) = \underbrace{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}_{n \times \sqrt{2}},$$

we see no so evident preference for $\lambda = 4$. So we choose $F(0) = 5$ for the super-exponential above 4. From (2.22), for $\eta = u$, we get

$$(4.9) \qquad F_{4,5}(z) \quad = \quad u(s_{4,5}e^{\kappa_4 z}) + 4$$

$$(4.10) \qquad s_{4,5} \quad = \quad U(5-4) = U(1) \approx 0.69470714714324 \ .$$

In analogy with v and V, let U denote the inverse function of u; and let $F_{4,5}$ denote the corresponding super-exponential. The choice $F_{4,5}(0) = 5$ allows the interpretation of $F_{4,5}(n)$ as result of n applications to number 5 of the exponential to base $\sqrt{2}$:

$$(4.11) \qquad F_{4,5}(n) = \exp_{\sqrt{2}}^{[n]}(5) = \underbrace{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}^5}}}}}_{n \text{ repetitions of } \sqrt{2}},$$
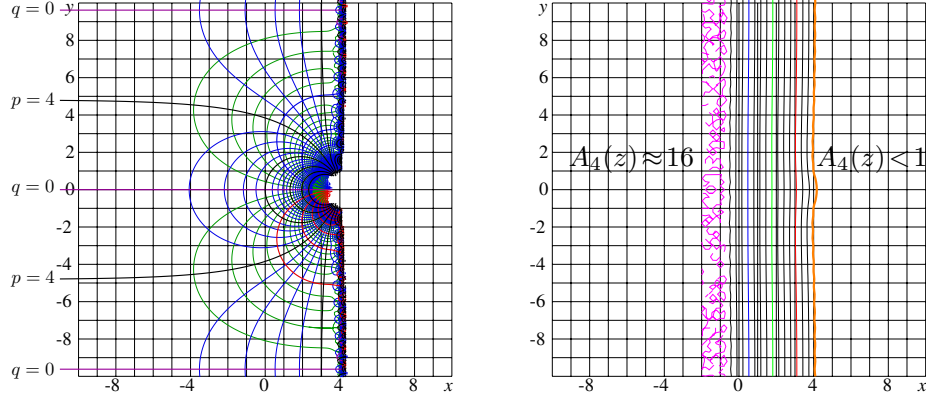
FIGURE 4. Left: Function $f = F_{4,5}(z)$ in the $x + iy$ plane; levels $p = \Re(f) =$ const and levels $q = \Im(f) =$ const are shown with thick lines for integer values of p and q. Right: Function $f = A_4(z)$ by (4.13) in the $x + iy$ plane; levels $f = 1, 4, 8, 12, 16$ are shown with thick lines.

at least for natural numbers n.

Note, that the super-exponentials below 2, for example $F_{2,1}$, remains smaller than 2 along the real axis, actually in the interval $(-2, \infty)$, as we did not define it for $(-\infty, -2]$; but the super-exponentials above 4, for example $F_{4,5}$, remains larger than 4 along the whole real axis. The super-exponential below 4 will be considered later together with the super-exponential above 2; along the real axis, these functions are also plotted in figure 3.

For the evaluation of (4.9) and (4.10), in analogy with super-exponential developed at $\lambda = 2$, we truncate the powerseries. The truncation of the powerseries u, keeping $N + 1$ terms, leads to the following approximation, let us call it $\tilde{F}_{4,5,N}$, of the super-function $F_{4,5}$:

$$(4.12) \qquad \tilde{F}_{4,5,N}(z) = 4 + \sum_{n=1}^{N} u_n s_{4,5}^n \exp(n \kappa_4 z) \; ;$$

The function $\tilde{F}_{4,5,20}$ is plotted in the complex z-plane in the left hand side picture of the figure 4. In the right part of the left picture, the density of the levels is so high, that the plotter cannot draw them; this part is left empty.

In order to characterize the precision of the approximation of solution $F_{4,5}$ of equation (1.1), in the right hand side picture of figure 4, the agreement

$$(4.13) \qquad A_4(z) = -\lg \left| \frac{\tilde{F}_{4,5,20}(z) - \exp_b\big(\tilde{F}_{4,5,20}(z-1)\big)}{\tilde{F}_{4,5,20}(z) + \exp_b\big(\tilde{F}_{4,5,20}(z-1)\big)} \right|$$

is plotted. The plot indicates, that, at $\Re(z) < -1$, the error of evaluation is determined by the rounding errors of the complex$\langle$double$\rangle$ arithmetic, not by the error of the approximation $\tilde{F}_{4,5,20}$. For the approximation of the solution $F_{4,5}$ in the whole complex plane, we suggest the function

$$(4.14) \qquad \tilde{F}_{4,5}(z) = \begin{cases} \tilde{F}_{4,5,N}(z) & , \quad \Re(z) < x_N \\ \exp_b\big(\tilde{F}_{4,5}(z-1)\big) & , \quad \Re(z) \geq x_N \end{cases}$$

For $N = 20$ and $x_{20} = -1$ it returns of order of 14 significant figures of function $F_{4,5}$. This approximation is used to plot $F_{4,5}$ in figures 2 and 3. The evaluation can be performed also at smaller values of N; then, for the good precision we need to choose $|x_N| > |x_{20}|$.

In the left hand side of the complex plane, $F_{4,5}$ approaches the fixed point $\lambda = 4$. Along the real axis, the function grows up faster than any exponential, and it shows the fractal behavior in vicinity of the positive part of the real axis. At moderated distances from the real axis, the function $F_{4,5}$ changes slowly, approaching the fixed point $\lambda = 2$.

The function $F_{4,5}$ is periodic; the period T_4 is expressed by

$$(4.15) \qquad T_4 = \frac{2\pi i}{\kappa_4} \approx 19.236149042042854712 \, i \, ,$$

so, at the distance of order of 19 from the real axis, the fractal structure is reproduced, although it is out of range of its plot in figure 2. The function shows rapid growth in the positive part of the real axis and complicated behavior in its vicinity, but it is analytic in every point of the complex plane:

Proposition 11. $F_{4,5}$ is entire.

Proof. $h = \exp_{\sqrt{2}}$ is entire and $K = 2\ln(2) > 1$. So proposition 4 can be applied. $\square$

However, the growth of the entire $F_{4,5}$ in vicinity of the real axis is so rapid, that even in the range of the plot in figure 2, the values cannot be stored in a complex$\langle$double$\rangle$ variable; there values are marked with symbol ∞. Such a rapid growth is typical for the growing super-exponentials.

4.1. **Inverse super-exponential above 4.** The inverse function $F_{4,5}^{-1}$ of function $F_{4,5}$ can be defined with relation

$$(4.16) \qquad F_{4,5}\big(F_{4,5}^{-1}(z)\big) \;=\; z \; \forall z \in \mathbb{C}\backslash\{x \in \mathbb{R} : x \le 4\} \, ,$$

$$(4.17) \qquad \left| \Im\big(F_{4,5}^{-1}(z)\big) \right| \;<\; \Im(T_4)/2 \approx 9.6180745210214273558$$

It is shown at the bottom of figure 2. The range of validity of the reciprocal relation

$$(4.18) \qquad F_{4,5}^{-1}\big(F_{4,5}(z)\big) \;=\; z$$

is narrower, than $\mathbb{C}\backslash\{x \in \mathbb{R} : x \le 4\}$; it is limited by a single period of function $F_{4,5}$ along the imaginary axis. The function $F_{4,5}^{-1}(z)$ has only two singularities at $z = 2$ and $z = 4$, we choose the cut line along the real axis, $z < 4$. In the positive direction of the real axis, the function slowly grows, slower than a logarithm. We consider the properties of this function and its evaluation in this section.

The inverse function satisfies the Abel equation [1, 3]

$$(4.19) \qquad F_{4,5}^{-1}\big(\exp_b(z)\big) = F_{4,5}^{-1}(z) + 1$$

This function can be expressed with the Schröder powerseries U of h[4]:

$$(4.20) \qquad F_{4,5}^{-1}(z) = \log(U(z-4)/s_{4,5})/\kappa_4$$

We compute the first some coefficients of U with formula (2.15)

$$
\begin{array}{lll}
U_1 = & 1 \\
(4.21) \quad U_2 = & -u_2 = \frac{\ln(2)/4}{1-2\ln(2)} & \approx -0.4485874311952612289 \\
U_3 = & \frac{(4\ln(2)+1)/24}{1-2\ln(2)-4\ln(2)+8\ln(2)^2} & \approx \ 0.21208912005491969757
\end{array}
$$

More values are printed in the second column of table 2. Numerically, the inverse function can be implemented with

$$
(4.22) \qquad \tilde{F}_{4,5,N}^{-1}(z) = \frac{1}{\kappa_4} \log \left(\frac{1}{s_{4,5}} \sum_{n=1}^{N} U_n \cdot (z-4)^n \right) \ ;
$$

this approximation is useful for small values of $|z-4|$. The range of the approximation can be extended with function

$$
(4.23) \qquad \tilde{F}_{4,5}^{-1}(z) = \left\{ \begin{array}{ll} \tilde{F}_{4,5}^{-1}\big(\log_b(z)\big)+1 & , \quad |z-4| > \rho \\ \tilde{F}_{4,5,N}^{-1}(z) & , \quad |z-4| \le \rho \end{array} \right.
$$

At $N = 20$ and $\rho = 0.4$, such approximation returns at least 14 significant figures and can be recommended for a complex$\langle$double$\rangle$ implementation.

5. SUPER-EXPONENTIALS ABOVE 2 AND BELOW 4

In this section, we consider the two remaining classes of super-exponentials to base $\sqrt{2}$. These are exemplary given by

$$
(5.1) \quad F_{2,3}(z) = v(s_{2,3}e^{\kappa_2 z})+2 \ , \quad s_{2,3} = V(3-2) = \ \ V(1) \approx \ \ 2.1844747586390
$$

$$
(5.2) \quad F_{4,3}(z) = u(s_{4,3}e^{\kappa_4 z})+4 \ , \quad s_{4,3} = U(3-4) = U(-1) \approx -1.8604041940071
$$

s is chosen such that $F_{2,3}(0) = F_{4,3}(0) = 3$.

Every regular super-exponential $F(z)$ at fixed point λ is a translation in the z-plane of the principal (see definition 3 and corollary 8) super-exponential. If s is positive it is a real translation otherwise it is a complex translation. The super-exponential with $s = -1$ can be obtained by a translation along the imaginary axis by the half of the corresponding period $T = 2\pi\mathrm{i}/\kappa$:

$$
-e^{\kappa z} = e^{\kappa z+\pi\mathrm{i}} = e^{\kappa(z+\pi\mathrm{i}/\kappa)} = e^{\kappa(z+T/2)}.
$$

For systematization of our super-exponentials, let us begin with the case, when, in (2.22), the Schröder parameter s is unity. We call the corresponding superfunction "principal super-exponential" or, if we need to indicate value of κ, "the κ-principal super-exponential". Let $\mathcal{F}_\lambda$ denotes the principal super-exponential at λ, id est, $\mathcal{F}_\lambda(z) = \eta(\exp(\kappa_\lambda z)) + \lambda$; we use this expression for functions $\eta = u$ and $\eta = v$. Then, all our super-exponentials can be expressed as principal superexponentials with displaced arguments:

$$
(5.3) \qquad F_{2,1} = \mathcal{F}_2(z+t_{2,1}+T_2/2) \qquad t_{2,1} = \frac{\ln|s_{2,1}|}{\kappa_2} \approx \ \ 1.25155147882219
$$

$$
(5.4) \qquad F_{2,3} = \mathcal{F}_2(z+t_{2,3}) \qquad t_{2,3} = \frac{\ln(s_{2,3})}{\kappa_2} \approx -2.13191778709503
$$

$$
(5.5) \qquad F_{4,3} = \mathcal{F}_4(z+t_{4,3}+T_4/2) \qquad t_{4,3} = \frac{\ln|s_{4,3}|}{\kappa_4} \approx \ \ 1.90057764535872
$$

$$
(5.6) \qquad F_{4,5} = \mathcal{F}_4(z+t_{4,3}) \qquad t_{4,5} = \frac{\ln(s_{4,5})}{\kappa_4} \approx -1.11520724513161
$$
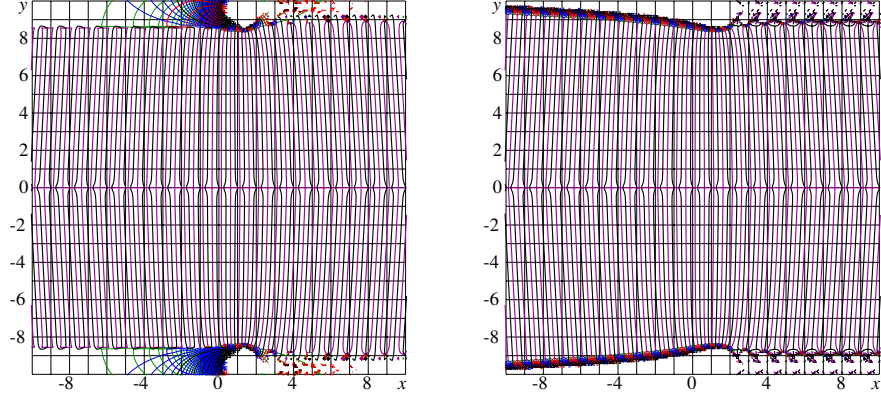
FIGURE 5. Functions w and $\tilde{w}$ by (5.11) in the complex plane

In consequence the functions $F_{2,3}$ and $F_{4,3}$ correspond to a complex translation of the functions $F_{2,1}$ and $F_{4,5}$, respectively:

$$(5.7) \qquad F_{2,3}(z) = F_{2,1}(z+t_{2,3}-t_{2,1}-T_2/2)$$

$$(5.8) \qquad F_{4,3}(z) = F_{4,5}(z+t_{4,3}-t_{4,5}-T_4/2)$$

We place the cut lines of $F_{2,3}^{-1}$ and $F_{4,3}^{-1}$ along the half-lines line $\Re(z) < 2$ and $\Re(z) > 4$, in order to keep them analytic in vicinity of $z = 3$. $F_{2,3}$ and $F_{4,3}$ are shown in the left hand side of figure 2 and their inverse are plotted in the right hand side.

The plots for $F_{2,3}$ and $F_{4,3}$ look similar. At least for $\Im(z) < T_2/2$, the two $(0 \mapsto 3)$ super-exponentials are related with

$$(5.9) \qquad F_{2,3}(z) = F_{4,3}\big(z + \theta_{2,4}(z)\big) , \qquad \theta_{2,4}(z) = F_{4,3}^{-1}\big(F_{2,3}(z)\big) - z ,$$

where $\theta_{2,4}$ is a 1-periodic function. At least in vicinity of the real axis, it can be approximated with the sinusoidal

$$(5.10) \qquad \theta_{2,4}(z) \approx \tilde{\theta}_{2,4}(z) = \xi \cdot \big(\sin(\phi+2\pi z) - \sin(\phi) \big)$$

with parameters $\xi = 0.7252175546 \cdot 10^{-24}$ and $\phi = 0.74633983476$. The approximation $\tilde{\theta}_{2,4}$ can be used for the complex⟨double⟩ implementation of $w(z) = F_{4,3}(z) - F_{2,3}(z)$ for $|\Im(z)| < 4$:

$$(5.11) \qquad w(z) = F_{4,3}(z) - F_{2,3}(z) \approx \tilde{w}(z) := \tilde{\theta}_{24}(z) \cdot F_{4,3}'(z)$$

Function w is plotted in the left hand picture of figure 5, and function $\tilde{w}$ in the right hand picture. The plots of w and $\tilde{w}$ look identical at moderate values of the imaginary part of the argument. In addition, each of them is zero, while the argument is integer. However, the entire function $\tilde{w}$ reproduces neither the cutlines, nor singularities of function w; and the deviation of $\tilde{w}$ from w becomes visible at the distance of order of $|T_2|/2 \approx 8.5$ form the real axis.

Any small periodic or quasi-periodic perturbation of a holomorphic function, real along the real axis, grows up in the direction of imaginary axis. The quasi-period unity determines the increment of this growth; is of order of 2π. In the case of function w, this perturbation becomes of order of unity at the distance of order of $|T_2|/2$. This allows to estimate the smallness of the deviation w in vicinity of

the real axis. At the distance of order $|T_2|/2$ from the real axis, the quasi-periodic perturbation of the function grows up until unity (and even becomes infinite at the singularities of $F_{2,3}$). This gives an estimate of the order of magnitude of this deviation; for moderate values of $\Im(z)$, id est, in vicinity of the real axis,

$$(5.12) \qquad |w(z)| \;<\; \sigma \;:=\; \exp(-\pi\,|T_2|) = \exp\!\left(\frac{-2\pi^2}{|\ln(\ln(2))|}\right)$$
$$\approx\;\; \exp(-53.8) \approx 4\cdot 10^{-24}$$

The perturbation is not exactly sinusoidal. As we leave from the real axis, the highest harmonics grow faster; so, we may expect, that, at the real axis, the deviation is smaller than σ. The function $10^{24}w(x)$ versus real x is shown at the bottom of figure 3; it oscillates in vicinity of the real axis and, indeed, for real x the values of $|w(x)|$ are an order of magnitude smaller than σ.

In such a way, the order of magnitude of the deviation between the two $(C,\ 0\mapsto 3)$ super-functions of $\exp_b$ is determined by their periods, which, in turn, is determined by the derivatives of the exponentials at the fixed points $\lambda=4$ and $\lambda=2$ for $b=\sqrt{2}$.

The deviation $\mathcal{D}$ by (7.3) of the iterated exponentials, made of super-function with periods T_4 and T_2 also is smaller than σ in the (2,4) interval; this deviation is considered in section 7. The deviation between two holomorphic functions cannot remain small everywhere (while this deviation is not a constant); therefore it can be seen comparing the maps of these functions in the complex plane. The extension to the complex plane is a universal tool for the analysis of functions that differ only little at the real axis.

6. PERIODICITY OF INVERSE SUPER-EXPONENTIALS

When considering inverse holomorphic super-functions in the complex plane one is tempted to make some mistakes of the same category as the branch-related mistake $\log(ab)=\log(a)+\log(b)$ for $a,b\in\mathbb{C}\setminus\{0\}$. One merely can state that for each a and b we can choose a branch of the logarithm on the left side such that the previous equation is true. Or one can restrict the validity of the equation to the right half-plane.

Without going too deep into the theory of global holomorphic functions, we want to mention this source of confusion. Especially for derivations like:

$$(6.1) \qquad\qquad F(z+1) = h\big(F(z)\big)$$
$$(6.2) \qquad\qquad F^{-1}(z) + 1 = F^{-1}\big(h(z)\big)$$

the inverse of a super-function may be an Abel-function only on some restricted domain or up to choosing the correct branch.

Moreover for inverse real-analytic super-exponentials G the following derivation is correct only up to choosing the proper branch. Let $T = 2\pi\mathrm{i}/\ln(b)$ be the period of $z\mapsto b^z$:

$$(6.3) \qquad\qquad G(b^z) = G(z) + 1$$
$$(6.4) \qquad\quad G(z) + 1 = G(b^z) = G\big(b^{z+T}\big) = G(z+T) + 1$$
$$(6.5) \qquad\qquad\qquad G(z) = G(z+T)$$

Proposition 12. *Let G be the inverse (taken along the real axis and analytically continued) of a regular super-exponential to base $b \in (1, e^{1/e})$, then G is periodic*

$$(6.6) \qquad G(z+T) = G(z) , \qquad\qquad T = \frac{2\pi i}{\ln(b)}$$

up to choosing a suitable branch of G.

From the considered inverse super-exponentials — with cuts as indicated — the only properly periodic function is $F_{2,1}^{-1}$, i.e.

$$(6.7) \quad F_{2,1}^{-1}(z+T_{\text{slog}}) = F_{2,1}^{-1}(z) , \qquad T_{\text{slog}} = \frac{4\pi i}{\ln(2)} \approx 18.129440567308775239\ i$$

for each $z \neq (2, \infty) + T_{\text{slog}}k$, $k \in \mathbb{Z}$.

The period T_{slog} is determined by the jump of the function $F_{2,1}$ at the cut between points -1 and -2; which is of the same order of magnitude as the period of the super-exponential $F_{4,5}$ by (4.15) and the period T_2 of the tetrational by (3.15). The closeness of the periods T_2 to the period T_{slog} seems to be a specific property of the tetrational to base $\sqrt{2}$. In particular, such a closeness allows functions $F_{2,1}$ and $x \mapsto -F_{2,1}^{-1}(-x)$ to be nearly identical along the real axis; this topic is considered in section 8.

7. Non-integer iteration of the exponential to base sqrt(2)

The pair $F_{\lambda,d}$ and $F_{\lambda,d}^{-1}$ can be used to define fractional iteration of the exponentiation to the base $b = \sqrt{2}$. For $\lambda = 4$, we define

$$(7.1) \qquad \exp_{b,4}^{[c]}(z) = F_{4,d}\left(c + F_{4,d}^{-1}(z)\right) \forall z \in \mathbb{C} \backslash \{x \in \mathbb{R} : x \leq 4\} .$$

In particular, we may use $d=3$ or $d=5$ in (7.1).

Due to relation (5.8), the displacement of the argument of function $F_{\lambda,d}$ is compensated by the displacement of value of of function $F_{\lambda,d}^{-1}$. For this reason, we do not specify the value of d in the left hand side of (7.1); it does not depend on d.

For $c = 0.5$ the $\exp_{b,4}^{[c]}$ by (7.1) is shown in the right hand side picture of figure 6 in the same notations as in figure 1.

However, the shape of the iterated exponential depends on the fixed point λ used to construct the super-functions. The pair of functions $F_{2,d}$ and $F_{2,d}^{-1}$ also allows to construct the iterated exponential

$$(7.2) \qquad\qquad \exp_{b,2}^{[c]}(z) = F_{2,d}(c + F_{2,d}^{-1}(z))$$

in a similar way as is constructed with pair $F_{4,d}$ and $F_{4,d}^{-1}$ in equation (7.1). The range of holomorphism is a little bit more complicated than in the case $\lambda = 4$ because of the complicated behavior of the $F_{2,d}^{-1}$ shown in the right hand side of figure 2. Similarly, due to relation (5.7), we can use any of the already described functions, either $d=1$ or $d=3$, and get the same $\exp_{b,2}^{[c]}$. For $c = 0.5$, this function is plotted in the left hand side picture of figure 6.

The functions $\exp_{\sqrt{2},2}^{[c]}$ and $\exp_{\sqrt{2},4}^{[c]}$ versus real argument are plotted in the bottom, right picture of figure 6. In the range $2 < x < 4$ these curves overlap. (They were expected to overlap, because, along the real axis, the functions $F_{2,3}$ and $F_{4,3}$ overlap with 24 significant figures, see figure 3).
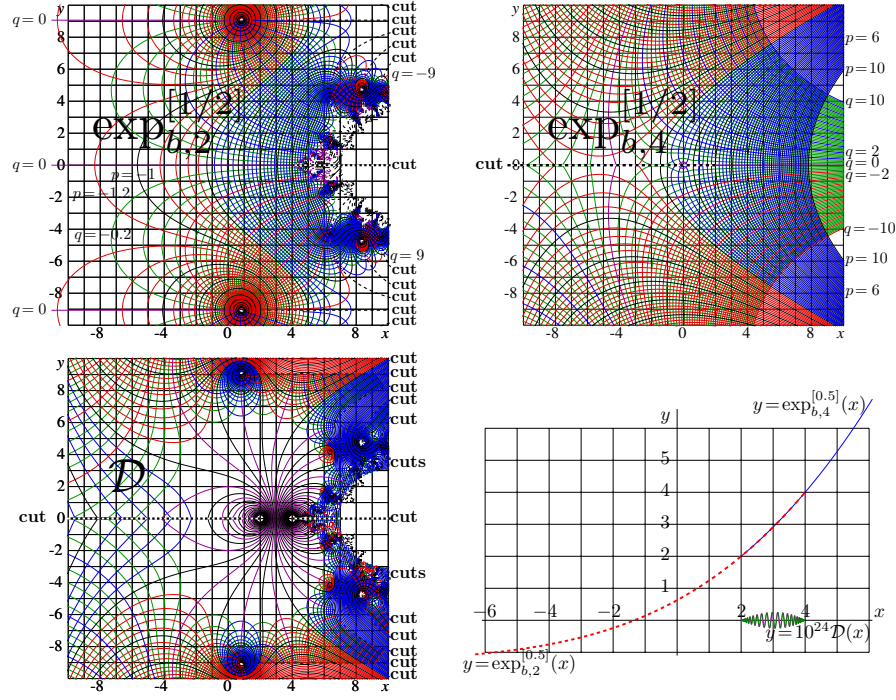
FIGURE 6. Comparison of functions $\exp_{b,4}^{[c]}$ (top, left) and $\exp_{b,2}^{[c]}$ (top, right) for $c = 0.5$; difference $\mathcal{D}$ (left,bottom) in the complex plane; and both functions versus real argument (right, bottom): $y = \exp_{b,4}^{[c]}(x)$ by 7.1, solid, and $y = \exp_{b,2}^{[c]}(x)$ by 7.2, dashed; function $\mathcal{D}$ by (7.3), scaled with factor 10^{24}, oscillates in vicinity of the abscissa axis, $2 < x < 4$.

In spite of the similar values along the real axis, functions $\exp_{\sqrt{2},4}^{[0.5]}$ and $\exp_{\sqrt{2},2}^{[0.5]}$ are different. Function $\exp_{b,4}^{[c]}(z)$ has a single cut-line at $z < 2$. Function $\exp_{b,2}^{[c]}(z)$ has two sets of cut-lines

$$\Re(z) > 4 \ , \ \Im(z) = \Im(T_{\text{slog}})n \ , \ n \in \mathbb{N};$$

and

$$\Re(z) > \Re\left(F_{2,1}\left(-\frac{5}{2} \pm o\mathrm{i}\right)\right) \approx 0.8386449382477 \ , \ \Im(z) = \Im\left(T_{\text{slog}}\right)\frac{2n+1}{2} \ , \ n \in \mathbb{N};$$

also, this function has a fractal of singularities and additional cut-lines at $\Re(z) > 4$, due to the singular behavior of function $F_{4,5}^{-1}(z)$ in that region.

Function $\exp_{b,4}^{[c]}(z)$ is not periodic, as $F_{4,5}^{-1}$ is not periodic.

Function $\exp_{b,2}^{[c]}(z)$ is periodic, as $F_{2,1}^{-1}$ is periodic; the period T_{slog} is determined by (6.7).

In order to characterize the deviation of the two iterated exponentials, we analyze the difference

$$(7.3) \qquad\qquad \mathcal{D}(z) = \exp_{\sqrt{2},4}^{[0.5]}(z) - \exp_{\sqrt{2},2}^{[0.5]}(z) \ .$$

Function $\mathcal{D}$, scaled with factor 10^{24} is plotted in the range $2 < x < 4$ in the right bottom part of figure 6; $|\mathcal{D}(x)| < \sigma$.

The two iterated exponentials strongly deviate from each other in vicinity of $z = \Re(F_{2,1}(-2.5 + oi)) + \mathrm{T}_{\mathrm{slog}}/2$, where the function $\exp_{b,2}^{[c]}(z)$ has a singularity. (Symbol oi indicates that the function should be evaluated just above the cutline.)

For $|z-3| < 1$, the function $\mathcal{D}(z)$ can be roughly approximated with the 6-parametric fit

$$
(7.4) \quad
\begin{aligned}
\tilde{\mathcal{D}}(z) = {} & 2.48 \cdot 10^{-25}(z-2)(4-z)\Big(1 + 0.120(z-3) + 0.006(z-3)^2\Big) \times \\
& \sin\Big(0.747 - 0.068(z-3) + 0.007(z-3)^2 + T_4\ln(4-z) + T_2\ln(z-2)\Big)
\end{aligned}
$$

The constants T_2 and T_4 are determined by (3.15) and (4.15). In figure 7, both functions $10^{24}\mathcal{D}$ and $10^{24}\tilde{\mathcal{D}}$ overlap well and the error of the approximation $\tilde{\mathcal{D}}$ is not seen even at strong zooming-in. For detailed numerical analysis of the deviation between the two $(0 \mapsto 3)$ super-exponentials in vicinity of the real axis, even more precise fit of function $\mathcal{D}$ can be elaborated on the base of approximation (5.11) of difference w between the two super-exponentials, treating the constant σ by (5.13) as a small parameter.

As in the case of super-exponentials $F_{2,3}$ and $F_{4,3}$ and as in the case of the iterated exponentials $\exp_{b,2}^{[c]}$ and $\exp_{b,4}^{[c]}$ precise evaluation is required to see that the two functions are not the same. Dan Asimov privately communicated that he, Dean Hickerson and Richard Schroeppel recognized this deviation of the half-iterate around 1991 by computing the values to 25 digits of accuracy. The only accessible reference is his posting [8] on sci.math in form of an April fool's joke.

However, the deviation becomes more visible outside the real axis, functions are clearly distinguishable without any zooming-in. In the following section we consider an additional (and last) example when leaving the real axis significantly simplifies the consideration.

8. About the apparent (x,-y) symmetry of the graph of the super-exponential below 2.

The $(x, -y)$ symmetry of graph of $y = f(x)$, apparently shown in figure 7 for $f = F_{2,1}$, can be described by the formula

$$(8.1) \qquad\qquad f^{-1}(-x) = -f(x)$$

or, applying function f to equation (8.1),

$$(8.2) \qquad\qquad f\big(-f(x)\big) = -x \ .$$

Generally, for some function f, there is nothing wrong in relations (8.1), (8.2). For example, each of functions

$$f(x) = 1 + x \ , \qquad f(x) = \sqrt{25 - (3-x)^2} - 3 \ , \qquad f(x) = \frac{2x+3}{x+2}$$

satisfies (8.1), (8.2); and the graphic of the last of them even has the same vertical and horizontal asymptote $x = -2$ and $y = 2$ (although does not approach them exponentially). In this section we show, that function f, satisfying (8.2), (8.1), cannot be $F_{2,1}$.
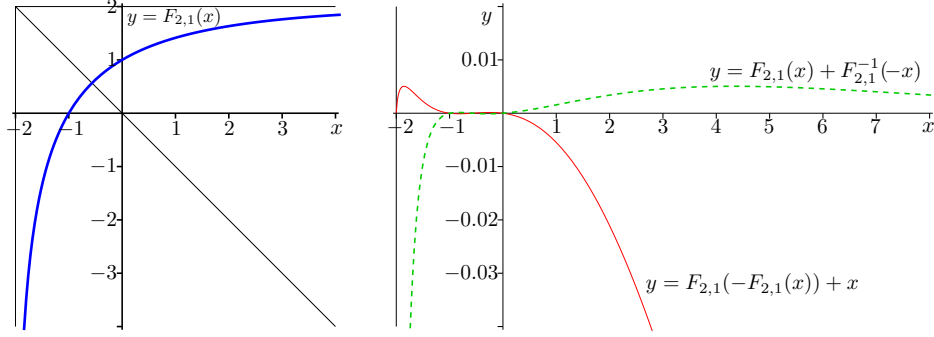
FIGURE 7. Function $y = F_{2,1}(x)$ of real argument left; functions $y = F_{2,1}(-F_{2,1}(x)) + x$ and $y = F_{2,1}(x) + F_{2,1}^{-1}(-x))$, right.

Below, we present two independent proofs. The first proof shows, that such a symmetry does not hold for a function with asymptotic behavior of function $F_{2,1}$. The second proof shows more generally, that such a symmetry cannot hold for any super-exponential f to base $\sqrt{2}$ that is monotonous on the $(-2, \infty)$ interval. Both propositions below and their proofs operate exclusively with real quantities.

Proposition 13. $y = F_{2,1}(x)$ is not $(x, -y)$ symmetric.

Proof. Here, we do not put indices on the function f, assuming that $f = F_{2,1}$. We show that for such function, equation (8.2) does not hold. The deduction is based on the asymptotic properties of the function. Consider the behavior of function $f(x)$ in vicinity of the singularity at $x = -2$:

$$(8.3) \qquad f(-2+x) = \log_b(f(-1+x)) = \log_b\Big(f'(-1)x + \mathcal{O}(x^2)\Big)$$

In the right hand side of (8.3), the expansion of f begins with linear term, because $f(-1) = 0$. Using notation

$$(8.4) \qquad\qquad \Phi := f'(-1) = \frac{F_{2,1}{}'(0)}{\ln\left(\sqrt{2}\right)} \approx 1.69973893 \ ,$$

equation (8.3) can be rewritten as

$$(8.5) \qquad f(-2+x) = \log_b\big(\Phi x + \mathcal{O}(x^2)\big) = \log_b\Big((\Phi x) \cdot \big(1 + \mathcal{O}(x)\big)\Big)$$

For positive u, v, $\ln(uv) = \ln(u) + \ln(v)$; so, at least for positive x,

$$
\begin{aligned}
f(-2+x) &= \log_b(\Phi) + \log_b(x) + \log_b\big(1 + \mathcal{O}(x)\big)\\
(8.6) \qquad &= \log_b(x) + \log_b(\Phi) + \mathcal{O}(x) \ .
\end{aligned}
$$

Now remember the asymptotic behavior of $F_{2,1}(z)$ at large values of $\Re(z)$; from (3.9) and (3.3), we can represent $f(x)$ as follows:

$$(8.7) \qquad\qquad f(x) = 2 - \omega - \mathcal{O}(\omega^2) \ ,$$

where $\omega = -s_{2,1}e^{\kappa_2 x} > 0$. Then, consider the expansion at large positive x of the expression

$$(8.8) \qquad f\big(-f(x)\big) = f\Big(-2 + \omega + \mathcal{O}(\omega^2)\Big) \ .$$

Using the expansion (8.6) at $x = \omega + \mathcal{O}(\omega^2)$, we get

$$f\big(-f(x)\big) \quad = \quad \log_b\Big(\omega \cdot (1 + \mathcal{O}(\omega))\Big) + \log_b(\Phi) + \mathcal{O}(\omega)$$

(8.9)
$$= \quad \log_b(\omega) + \mathcal{O}(\omega^0)$$

Substituting ω to $\exp\big((x + \ln|s_{2,1}|)\kappa_2\big)$, for $x \to \infty$ we get

(8.10) $$f\big(-f(x)\big) = \frac{\kappa_2 x}{\ln(\sqrt{2})} + \mathcal{O}(x^0) = \frac{2\ln(\ln(2))}{\ln(2)} x + \mathcal{O}(x^0) \ ;$$

equation (3.2) is used to get the last equality. The coefficient

(8.11) $$\frac{2\ln(\ln(2))}{\ln(2)} \approx -1.057532745889795$$

is close to minus unity, but not exactly minus unity. At large positive x, the expression $-f\big(-f(x)\big)$ deviates from x asymptotically linearly, with tangent of order of 6%. This deviation is seen at the plot of $y = f(-f(x)) + x$ in the right hand picture of figure 7. $\qquad\square$

The symmetry can be negated also without to refer the asymptotic analysis, just from the observation that f is a monotonous (increasing or decreasing) function. The deduction based on the sequential application of hypothesis (8.2) is presented below.

Proposition 14. *There is no monotonous function f on $(-2, \infty)$ that satisfies*

(8.12) $$f(0) = 1$$

(8.13) $$f(x+1) = \sqrt{2}^{f(x)}$$

(8.14) $$f(-f(x)) = -x$$

Proof. Below we show that there exist 3 points x_1, x_2, x_3 such that $x_1 < x_2 < x_3$, but $f(x_1) < f(x_2) > f(x_3)$. No one monotonous function is allowed to have such a property.

Let us show that there are these 3 points $x_1 < x_2 < x_3$; set $b = \sqrt{2}$. We start with the more difficult part of finding $x_2 < x_3$ with $f(x_2) > f(x_3)$. Applying (8.13), we get

$$f(0) = 1$$
$$f(1) = b$$
$$f(2) = b^b =: c.$$

It follows further

$$f(4) = b^{b^c}$$
$$f(-b^{b^c}) = -4$$
$$f(-b^{b^c} + 1) = b^{-4}$$

and we set

(8.15) $$x_2 := -b^{-4} \qquad\qquad f(x_2) = b^{b^c} - 1.$$

On the other hand we conclude

$$f(-c) = -2$$
$$f(-c + 1) = b^{-2}$$
$$f(-b^{-2}) = c - 1$$
$$f(-b^{-2} + 1) = b^{c-1}$$
$$f(-b^{c-1}) = b^{-2} - 1$$

and set

$$(8.16) \qquad x_3 = -b^{c-1} + 1 \qquad\qquad f(x_3) = b^{b^{-2}-1}.$$

Now we compare x_2 and x_3

$$x_2 = -b^{-4} = -0.25 < -0.24510159 \approx -b^{c-1} + 1 = x_3$$

and their values

$$f(x_2) = b^{b^c} - 1 \approx 0.84091087 > 0.84089642 \approx b^{b^{-2}-1} = f(x_3)$$

x_1 can then easily be found: say $x_1 = -1 < x_2$ then $f(-1) = 0 < f(x_2)$.   □

There is no need to depart from the real axis in order to see that the $(x, -y)$ symmetry of the graphic in the left hand side picture of the figure 7 is only approximation, although in the range $(-1, 0)$, this "symmetry" holds with 4 decimal digits. Having the complex⟨double⟩ implementation of function f, the left hand side of the expression (8.2) can be precisely evaluated; the residual is plotted in the right hand picture of figure 7, confirming the approximate character of the $(x, -y)$ "symmetry".

We included both propositions above, because they are beautiful and could be good exercises for the first-grad students who do not yet deal with functions of complex variable. However, the approximate character of the $(x, -y)$ "symmetry" follows directly from the periodicity of functions $F_{2,1}$ and $F_{2,1}^{-1}$. Both functions are periodic, but their periods T_2 by (3.15) and T_{slog} by (6.7) are not equivalent. If two functions, holomorphic at some domain C, coincide at some interval, for example, $(-1, 0)$, then they coincide in the whole domain, and cannot have so different periods, assuming that they are not constant.

We have considered three pairs of "similar" functions: $(F_{2,3}, F_{4,3})$ by equations (5.1),(5.2), shown in figure 3; $\left(\exp_{\sqrt{2},2}^{[0.5]}, \exp_{\sqrt{2},4}^{[0.5]}\right)$ by equations (7.2),(7.1), shown in figure 6, and, en fin, $\left(F_{2,1}, x \mapsto -F_{2,1}^{-1}(-x)\right)$ by equations (3.9),(3.16), plotted in the left hand side picture of figure 7. Here, the "similarity" means, that at the real-real plot, visually we cannot distinguish the function from its partner. In the last case (considered in this section), the difference between these functions is plotted with dashed curve in the right hand side picture of figure 7; the float precision of evaluation is sufficient to reveal the deviation. In the first two cases, of order of 30 decimal digits should be evaluated in order to reveal the deviation along the real axis. In all examples considered, a high precision of evaluation is not required to see the deviation in the complex plane.

The different periodicity (in our case, periods T_2, T_{slog}, T_4 and ∞) allows to reject the hypothesis of the coincidence of the two functions without to make any numerical evaluation, without the asymptotic analysis of proposition 13, and without the algebraic deduction of proposition 14.

9. SUMMARY AND PROSPECTS

I. We have constructed four real super-functions $F_{\lambda,d}$ of the base-function $\exp_{\sqrt{2}}$ by (5.3)-(5.6). All these functions are periodic, their periods T_λ are pure imaginary and are determined by the fixed points $\lambda = 2$ and $\lambda = 4$. $F_{2,1}$ by equation (5.3) is a tetrational. The functions are bijective on the real axis with the following (co)domains (see figure 3):

$$F_{2,1}\colon (-2,\infty) \to (-\infty, 2) \qquad \text{strictly increasing}$$
$$F_{2,3}\colon (-\infty,\infty) \to (2, 4) \qquad \text{strictly decreasing}$$
$$F_{4,3}\colon (-\infty,\infty) \to (2, 4) \qquad \text{strictly decreasing}$$
$$F_{4,5}\colon (-\infty,\infty) \to (4, \infty) \qquad \text{strictly increasing}$$

II. The complex translations of the argument relates function $F_{2,1}$ to function $F_{2,3}$; both function are holomorphic at least in the right hand side of the complex plane. The complex translations of the argument relates function $F_{4,3}$ to function $F_{4,5}$; both functions are entire. We describe the complex $\langle$double$\rangle$ implementation for each of these functions and their inverses (see fig.2), and that for the deviation of $F_{4,5}$ from $F_{4,3}$, see figure 5. In vicinity of the real axis, this deviation is of order of 10^{-24}. Functions $F_{2,1}$ and its inverse function $F_{2,1}^{-1}$ have similar periods T_2 by (3.15) and T_{slog} by (6.7); in the range $-1 \le x \le 0$, the deviation of $-F_{2,1}^{-1}(-x)$ from $F_{2,1}(x)$ is of order of 10^{-4}, see figure 7.

III. With functions $F_{\lambda,d}$, the two different iterated exponentials $\exp_{b,2}^{[c]}$ and $\exp_{b,4}^{[c]}$ are constructed; they coincide at integer values of c, but for non-integer c, they have different asymptotic properties and different ranges of holomorphism. For $c = 0.5$ we plot both functions in figure 6; in the range $(2, 4)$ along the real axis, the deviation between these functions is of order of 10^{-24}, and each of these two functions can be arguably considered as a "true" $\sqrt{\exp_{\sqrt{2}}}$.

IV. The construction of the base-function h for a given super-function F is easier than the construction of a super-function for a given base-function. In principle the base-function is given by $h(z) = F(1 + F^{-1}(z))$. Such "inverse" method could be used to build-up some "table of super-functions" as an analogy of a table of integrals. Up to our knowledge, no such table for super-functions has been published. The compilation of such a table could be matter for future research.

V. The regular iteration of the base-function at a fixed point, can be used for construction of various real super-functions for real base-functions with real fixed points. For example, there should exist at least two different holomorphic half-iterates of the factorial (id est, $\sqrt{!}$); this function is used as logo of the Physics Department of the Moscow State University since past century, without to implement an algorithm for the efficient evaluation. Only recently an algorithm for evaluation of such a function has been suggested [13].

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Appendix A. Notations

symbol	description	ref.
A_2	agreement for complex(double) implementation of $F_{2,1}$	(3.12)
A_4	agreement for complex(double) implementation of $F_{4,5}$	(4.13)
a	reference point; for ex., $F(a) = d$	(1.2)
$b = \sqrt{2} \approx 1.414213562373$	base of the super-exponential	(1.4)
$\mathbb{C}$	set of complex numbers	
$C \subseteq \mathbb{C}$	range of the holomorphism of the superfunction F	(1.1)
c	number of iterations; $c = 0.5$ is used as example	(7.2)
$\chi = \eta^{-1}$	general symbol for Schröderfunction	(2.13)
$\mathcal{X}(z) = s\chi(x)$	any solution of (2.14) in Proposition 2	
$D \subseteq \mathbb{C}$	range of holomorphizm of function G	(2.3)
$\mathcal{D}(z) = \exp_{b,4}^{0.5} - \exp_{b,2}^{0.5}$	difference of two generalized exponentials	(7.3)
$\tilde{\mathcal{D}}(z)$	approximation of $\mathcal{D}(z)$ for $|z - 3| < 1$	(7.4)
d	variable used in definition of the super-function	(1.2)

symbol	description	ref.
$E(z) = F(z + \theta(z))$	super-function with modified argument	(1.3)
$\mathrm{e} = \exp(1) = \sum_{n=0}^{\infty} 1/n! \approx 2.71$	base of natural logarithms	
$\exp(z) = \mathrm{e}^z$	natural exponential	
$\exp_b(z) = b^z = \exp(\ln(b)z)$	exponential on the base b	(1.4)
$\exp_{b,\lambda}^c(z) = F_{\lambda,d}(c + F_{\lambda,d}^{-1}(z))$	iterated exponential developed at λ	(7.2)
$\mathcal{E}(z) = \mathrm{e}^{(z+t)\kappa}$	parameter to expand $F(z)$ in (2.2)	(2.4)
$\eta = \chi^{-1}$	inverse Schröder function; $F(z) = \eta(s\mathrm{e}^{\kappa z})$	(2.21)
η_n	general Taylor coefficient of the inverse Schröder	
F	super-function of base-function h, see (1.1) (2.22)	(1.4)
$F_{\lambda,d}$	$(C, 0 \mapsto d)$ super-function of $\exp_b$ developed at λ; for ex.,	(5.3)
$\mathcal{F}_\lambda$	principal super-function	(5.3)
$\Phi = F_{2,1}{}'(-1) = \frac{2}{\ln(2)} F_{2,1}{}'(0) \approx 1.69973893$	mathematical constant	(8.4)
$G(\mathcal{E}(z)) = F(z) - \lambda$	displaced super-function of modified argument	(2.2)
$g_n , \ n \in \mathbb{N}$	Taylor coefficients of function G at zero	(2.2)
$H_{k,x} = \{z/k \in \mathbb{C} : \Re(z) \leq x\}$	range of holomorphizm	Prop.7
h	base-function (or the transfer function)	(1.1)
h_n	for $n \in \mathbb{N}$, Taylor coefficients of h at λ	(2.1)
I	interval, where the regular superfunction is defined by	(2.24)
$i = \mathrm{i} = \sqrt{-1}$	imaginary unity	
$j \in \mathbb{R}, \ j > 0$	used in the example $h(z) = z^j$, $F(z) = \exp(j^z)$	
$K = K_\lambda = h_1 = h'(\lambda)$	derivative of base-function h at λ	(2.8)
$K_2 = h'(2) = \ln(2)$	derivative of h at the fixed point $\lambda = 2$	(3.1)
$K_4 = h'(4) = 2\ln(2)$	derivative of h at the fixed point $\lambda = 4$	(4.1)
$\kappa_\lambda = \ln(K_\lambda)$	wave-number	(3.2)
$\kappa_2 = \ln(\ln(2)) \approx -0.36651292058166432701$	wave-number for $\lambda = 2$	(3.2)
$\kappa_4 = \ln(2\ln(2)) \approx 0.32663425997828098238$	wave-number for $\lambda = 4$	(4.2)
λ	fixed point, solution of $h(\lambda) = \lambda$	
$\ln$	logarithm, inverse of exponentiation; $\ln = \log_e$; $\exp(\ln(z)) = z$	
m, n	used as integer variables	(2.1)
$\mathbb{N}$	set of natural numbers	
$\mathbb{R}$	set of real numbers	
O	grows up not faster (or decays not slower), than its argument	(8.9)
o	infinitezimal parameter, $F_{2,1}(-2.5 + o\mathrm{i})$ refers to upper manifold	
$\circ$	infix operation of combination of two functions	(2.17)
p, q	dependent variables, $\Re$ and $\Im$ parts of a function	fig.1
$\pi = 4\arctan(1) \approx 3.14$	$-\mathrm{i}/2$ of period of exp	(3.15)
r	various radii	
real function :	any f such that $f(z^*) = f(z)^*$ for all z where f is defined	
$s = K^t$	constant factor in the argument of the inverse Schröder	(2.22)
$s_{2,1} = V(-1) \approx -0.63209866105083$	Schröder parameter	(3.10)
$s_{2,3} = V(\ 1) \approx \ 2.1844747586390$	Schröder parameter	(5.1)
$s_{4,3} = U(-1) \approx -1.8604041940071$	Schröder parameter	(5.2)
$s_{4,5} = U(\ 1) \approx \ 0.69470714714324$	Schröder parameter	(4.10)
$\sigma = \exp(-2\pi^2/T_2) \approx 4 \cdot 10^{-24}$	upper bound for $|w(x)|$ at $x \in \mathbb{R}$	(5.13)
$T_2 = 2\pi\mathrm{i}/\kappa_2 \approx -17.143148179354847104 \, i$	period of $F_{2,1}$ and $F_{2,3}$	(3.15)
$T_4 = 2\pi\mathrm{i}/\kappa_4 \approx \ 19.236149042042854712 \, i$	period of $F_{4,5}$ and $F_{4,3}$	(3.15)

symbol	description	ref.
$T_{\mathrm{slog}} = \frac{4\pi\mathrm{i}}{\ln(2)} \approx 18.129440567308775239i$	period of $F_{2,1}^{-1}$, $F_{2,3}^{-1}$, $\exp_{b,2}^c$	(6.7)
t	displacement of argument of super-function	(2.4)
$t_{2,1} = \ln(s_{2,1})/\kappa_2 \approx 1.251551478822188$	displacement for $F_{2,1}(0)=1$	(5.3)
$t_{2,3} = \ln(s_{2,3})/\kappa_2 \approx 0.880366308272851$	displacement for $F_{2,3}(0)=3$	(5.4)
$t_{4,3} = \ln(s_{4,3})/\kappa_4 \approx 1.90057764535874$	displacement for $F_{4,3}(0)=3$	(5.5)
$t_{4,5} = \ln(s_{4,5})/\kappa_4 \approx 1.11520724513161$	displacement for $F_{4,3}(0)=5$	(5.6)
θ	modification of argument of super-function	(1.3)
$\theta_{2,4}(z) = F_{4,3}^{-1}\big(F_{2,3}(z)\big) - z \approx \tilde{\theta}_{2,4}(z)$		(5.10)
$\tilde{\theta}_{2,4}(z) = \xi \cdot \big(\sin(\phi+2\pi z) - \sin(\phi)\big)$	approximation of $\theta_{24}(z)$	(5.10)
$\xi = 0.7252175546 \cdot 10^{-24} \approx 0.2\ \sigma$	parameter in approximation $\tilde{\theta}_{24}$	(5.10)
$\phi = 0.74633983476$	parameter in approximation $\tilde{\theta}_{24}$	(5.10)
u	inverse Schröder function for $\lambda=4$	(4.9)
u_n, U_n	Taylor coefficients of G and G^{-1} for $\lambda=4$	table 2
v	inverse Schröder function for $\lambda=2$	(3.9)
v_n, V_n	Taylor coefficients of G and G^{-1} for $\lambda=2$	table 1
$w(z) = F_{4,3}(z) - F_{2,3}(z)$	deviation between two super-exponentials	(5.11)
$\omega = \mathcal{E}(z)$	small parameter used to expand function G	(2.7)
x, y	used as independent real variables	Fig.1
z	complex variable; sometimes, $z = x+\mathrm{i}y$	Fig.1
$\mathbb{Z}$	set of integer numbers	

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